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ANALYTIC TORSION FOR TWISTED DE RHAM COMPLEXES

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Abstract. We define analytic torsion $\tau(X,\mathcal{E},H)\in\det H^\bullet(X,\mathcal{E},H)$ for the twisted de Rham complex, consisting of the spaces of differential forms on a compact oriented Riemannian manifold X valued in a flat vector bundle $\mathcal{E}$, with a differential given by $\nabla^{\mathcal{E}}+H\wedge\cdot$, where $\nabla^{\mathcal{E}}$ is a flat connection on $\mathcal{E}$, H is an odd-degree closed differential form on X, and $H^\bullet(X,\mathcal{E},H)$ denotes the cohomology of this $\mathbb{Z}_2$-graded complex. The definition uses pseudo-differential operators and residue traces. We show that when $\dim X$ is odd, $\tau(X,\mathcal{E},H)$ is independent of the choice of metrics on X and $\mathcal{E}$ and of the representative H in the cohomology class $[H]$. We explain the relation to generalized geometry when H is a 3-form. We demonstrate some basic functorial properties. When H is a top-degree form, we compute the torsion, define its simplicial counterpart and prove an analogue of the Cheeger-Müller Theorem. We also study the twisted analytic torsion for T-dual circle bundles with integral 3-form fluxes.

Introduction

Let X be a compact closed oriented smooth manifold and $\rho\colon\pi_1(X)\to\mathrm{GL}(E)$, an orthogonal or unitary representation of the fundamental group $\pi_1(X)$ on a vector space E. The Reidemeister-Franz torsion, or R-torsion, of ρ is defined in terms of a triangulation of X. In [47, 48], Ray and Singer introduced an analytic counterpart as the alternating product of the regularized determinants of Laplacians and conjectured that it is equal to the R-torsion. (For lens spaces, the equality of the two torsions was established in [46].) The Ray-Singer conjecture was proved independently by Cheeger [19] and Müller [40] for orthogonal or unitary representations of the fundamental group and was extended to unimodular representations by Müller [41]. Another proof of the Cheeger-Müller theorem, as well as an extension of it to arbitrary flat bundles, is due to Bismut and Zhang [8], who used the Witten deformation technique.

In this paper we generalize the classical construction of the Ray-Singer torsion to the twisted de Rham complex with an odd-degree differential form as flux and with coefficients in a flat vector bundle. The twisted de Rham complex was first defined for twists by 3-form fluxes by Rohm and Witten in the appendix of [50] and has played an important role in string theory [11, 1], for the Ramond-Ramond fields (and their charges) in type II string theories lie in the twisted cohomology of spacetime. T-duality in type II string theories on compactified spacetime gives rise to a duality isomorphism of twisted cohomology groups [12].

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Let H be a closed differential form of odd degree on X and $\rho\colon \pi_1(X) \to \mathrm{GL}(E)$, a representation of $\pi_1(X)$ on a finite dimensional vector space E. Denote by $\mathcal{E}$ the corresponding flat bundle over X with the canonical flat connection $\nabla^{\mathcal{E}}$. The twisted de Rham complex is the $\mathbb{Z}_2$-graded complex $(\Omega^\bullet(X,\mathcal{E}), \nabla^{\mathcal{E}} + H\wedge\cdot)$. Its cohomology, denoted by $H^\bullet(X,\mathcal{E},H)$, is called the twisted de Rham cohomology. We show that the twisted cohomology groups are invariant under scalings of H provided its degree is at least 3 and under smooth homotopy equivalences that match the cohomology classes of the flux forms. We will define analytic torsion of the twisted de Rham complex $\tau(X,\mathcal{E},H) \in \det H^\bullet(X,\mathcal{E},H)$ as a ratio of zeta-function regularized determinants of partial Laplacians, multiplied by the ratio of volume elements of the cohomology groups. While the de Rham complex has a $\mathbb{Z}$-grading, the twisted de Rham complex is only $\mathbb{Z}_2$-graded. As a result, analytic techniques used to establish the basic properties in the classical case have to be generalized accordingly. These regularized determinants turn out to be more complicated to define, as they require properties of pseudodifferential projections. We show that when $\dim X$ is odd, $\tau(X,\mathcal{E},H)$ is independent of the choice of the Riemannian metric on X and the Hermitian metric on $\mathcal{E}$. The torsion $\tau(X,\mathcal{E},H)$ is also invariant (under a natural identification) if H is deformed within its cohomology class. The comparison of the deformations of the metrics and of the flux leads naturally to the concept of generalized metric [29]. We establish some basic functorial properties of this torsion. We then compute the torsion for odd-dimensional manifolds with a top-degree flux form. The latter is especially useful for 3-manifolds and leads to a conjecture in the general case. When the degree of H is sufficiently high we introduce a combinatorial counterpart of $\tau(X,\mathcal{E},H)$ and show that they are equal when H is a top-degree form. Finally, if (X,H) and $(\widehat{X},\widehat{H})$ are T-dual circle bundles with background fluxes, then the T-duality isomorphism identifies the determinant lines $\det H^\bullet(X,H) \cong (\det H^\bullet(\widehat{X},\widehat{H}))^{-1}$. Under this identification, we relate the twisted torsions for 3-dimensional T-dual circle bundles with integral 3-form fluxes.

The outline of the paper is as follows. In §1, we set up the notation in the paper and review the twisted de Rham complex and its cohomology [50, 11] with an odd-degree closed differential form as flux and with coefficients in a flat vector bundle associated to a representation of the fundamental group. In §2, we introduce the key definition of the analytic torsion of the twisted de Rham complex. Here, the property of pseudodifferential projection plays an important role. In §3, we show that the twisted analytic torsion is independent of the metrics on the manifold and on the flat bundle. We also show that it depends on the flux only through its cohomology class. The relation to generalized geometry is then explored. In §4, we establish the basic functorial properties of the analytic torsion for the twisted de Rham complex. §5 contains calculations of analytic torsion for the twisted de Rham complex and a simplicial version of it under certain restrictions. In this special case, the analogue of the Cheeger-Müller theorem is established. Finally, we study the behavior of the twisted analytic torsion under T-duality for circle bundles with a closed 3-form as flux.

There is an extensive literature on the torsion of $\mathbb{Z}$-graded complexes. Analytic torsion has been studied for manifolds with boundary [34, 35, 55, 20, 16], for the Dolbeault complex [49, 3, 6], in the equivariant setting [34, 35, 9, 17, 4], and for fibrations [21, 36], where torsion forms [3, 5, 7, 37, 38] appear. The analytic torsion was also identified as the partition function of certain topological field theories and

was studied for arbitrary ($\mathbb{Z}$-graded) elliptic complexes [51]. Recently, refined and complex-valued analytic torsions have been introduced and studied [13, 14, 18, 54]. It is tempting to extend these developments to $\mathbb{Z}_2$-graded complexes. Until recently, there appears to be no simplicial analogue of the twisted de Rham complex, except in a special case in §5.2, since the cup product is in general not graded commutative at the level of the cochain complex. However, in some work in progress with Getzler, it appears that we have succeeded in defining a simplicial analogue of the twisted de Rham complex, thereby enabling us to also define a simplicial analogue of the twisted analytic torsion in the general case.

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1. Twisted de Rham complexes

To set up the notation in the paper, we review the twisted de Rham cohomology [50, 11] with an odd-degree flux form and with coefficients in a flat vector bundle. We show that the twisted cohomology does not change under the scalings of the flux form when its degree is at least 3. We also establish the homotopy invariance of these cohomology groups.

1.1. **Flat vector bundles, representations and Hermitian metrics.** Let X be a connected, closed, compact, oriented smooth manifold. Let $\rho\colon \pi_1(X) \to \mathrm{GL}(E)$ be a representation of the fundamental group $\pi_1(X)$ on a vector space E. The associated vector bundle $p\colon \mathcal{E} \to X$ is given by $\mathcal{E} = (E \times \widetilde{X})/\sim$, where $\widetilde{X}$ denotes the universal covering of X and $(v, x\gamma) \sim (\gamma v, x)$ for all $\gamma \in \pi_1(X)$, $x \in \widetilde{X}$ and $v \in E$. A smooth section s of $\mathcal{E}$ can be uniquely represented by a smooth equivariant map $\phi\colon \widetilde{X} \to E$, satisfying $\phi(x\gamma) = \gamma^{-1}\phi(x)$ for all $\gamma \in \pi_1(X)$ and $x \in \widetilde{X}$.

Given any vector bundle $p\colon \mathcal{E} \to X$ over X, denote by $\Omega^i(X, \mathcal{E})$ the space of smooth differential i-forms on X with values in $\mathcal{E}$. A *flat connection* on $\mathcal{E}$ is a linear map

$$\nabla^{\mathcal{E}}\colon \Omega^i(X, \mathcal{E}) \to \Omega^{i+1}(X, \mathcal{E})$$

such that

$$\nabla^{\mathcal{E}}(f\omega) = df \wedge \omega + f\, \nabla^{\mathcal{E}}\omega \qquad \text{and} \qquad (\nabla^{\mathcal{E}})^2 = 0$$

for any smooth function f on X and any $\omega \in \Omega^i(X, \mathcal{E})$. If the vector bundle $\mathcal{E}$ is associated with a representation ρ as in the previous paragraph, an element of $\Omega^\bullet(X, \mathcal{E})$ can be uniquely represented as a $\pi_1(X)$-invariant element in $E \otimes \Omega^\bullet(\widetilde{X})$. If $\omega \in \Omega^\bullet(\widetilde{X})$ and $v \in E$, then $v \otimes \omega$ is said to be $\pi_1(X)$-invariant if $\gamma v \otimes \gamma^*\omega = v \otimes \omega$ for all $\gamma \in \pi_1(X)$. On such a vector bundle, there is a *canonical flat connection* $\nabla^{\mathcal{E}}$ given by, under the above identification, $\nabla^{\mathcal{E}}(v \otimes \omega) = v \otimes d\omega$, where d is the exterior derivative on forms.

The usual wedge product on differential forms can be extended to

$$\wedge\colon \Omega^i(X) \otimes \Omega^j(X, \mathcal{E}) \to \Omega^{i+j}(X, \mathcal{E}).$$

Together with the evaluation map $\mathcal{E} \otimes \mathcal{E}^* \to \mathbb{C}$, we have another product

$$\wedge\colon \Omega^i(X, \mathcal{E}) \otimes \Omega^j(X, \mathcal{E}^*) \to \Omega^{i+j}(X).$$

A Riemannian metric g_X defines the Hodge star operator

$$*\colon \Omega^i(X, \mathcal{E}) \to \Omega^{n-i}(X, \mathcal{E}),$$

where $n = \dim X$. An Hermitian metric $g_{\mathcal{E}}$ on $\mathcal{E}$ determines an $\mathbb{R}$-linear bundle isomorphism $\sharp\colon \mathcal{E} \to \mathcal{E}^*$, which extends to an $\mathbb{R}$-linear isomorphism

$$\sharp\colon \Omega^i(X,\mathcal{E}) \to \Omega^i(X,\mathcal{E}^*).$$

One sets $\Gamma = *\,\sharp = \sharp\,*$ and for any $\omega, \omega' \in \Omega^i(X,\mathcal{E})$, let

$$(\omega,\omega') = \int_X \bar{\omega} \wedge \Gamma\,\omega'.$$

This makes each $\Omega^i(X,\mathcal{E})$, $0 \le i \le n$, a pre-Hilbert space.

When $\mathcal{E}$ is associated to an orthogonal or unitary representation ρ of $\pi_1(X)$, $g_{\mathcal{E}}$ can be chosen to be compatible with the canonical flat connection. This is not possible in general. We will not assume that ρ is unimodular except in §5, where we calculate the torsion and establish an simplicial analogue under special conditions.

1.2. **Twisted de Rham cohomology.** Given a flat vector bundle $p\colon \mathcal{E} \to X$ and an odd-degree, closed differential form H on X, we set $\Omega^{\bar{0}}(X,\mathcal{E}) := \Omega^{\mathrm{even}}(X,\mathcal{E})$, $\Omega^{\bar{1}}(X,\mathcal{E}) := \Omega^{\mathrm{odd}}(X,\mathcal{E})$ and $\nabla^{\mathcal{E},H} := \nabla^{\mathcal{E}} + H \wedge \cdot$. We are primarily interested in the case when H does not contain a 1-form component, which can be absorbed in the flat connection $\nabla^{\mathcal{E}}$. We define the *twisted de Rham cohomology groups of* $\mathcal{E}$ as the quotients

$$H^{\bar{k}}(X,\mathcal{E},H) = \frac{\ker\,(\nabla^{\mathcal{E},H}\colon \Omega^{\bar{k}}(X,\mathcal{E}) \to \Omega^{\overline{k+1}}(X,\mathcal{E}))}{\mathrm{im}\,(\nabla^{\mathcal{E},H}\colon \Omega^{\overline{k+1}}(X,\mathcal{E}) \to \Omega^{\bar{k}}(X,\mathcal{E}))}, \quad k = 0,1.$$

Here and below, the bar over an integer means taking the value modulo 2. The groups $H^{\bar{k}}(X,\mathcal{E},H)$ ($k = 0,1$) are manifestly independent of the choice of the Riemannian metric on X or the Hermitian metric on $\mathcal{E}$. The corresponding *twisted Betti numbers* are denoted by

$$b_{\bar{k}} = b_{\bar{k}}(X,\mathcal{E},H) := \dim H^{\bar{k}}(X,\mathcal{E},H), \quad k = 0,1.$$

Suppose H is replaced by $H' = H - dB$ for some $B \in \Omega^{\bar{0}}(X)$, then there is an isomorphism $\varepsilon_B := e^B \wedge \cdot : \Omega^\bullet(X,\mathcal{E}) \to \Omega^\bullet(X,\mathcal{E})$ satisfying

$$\varepsilon_B \circ \nabla^{\mathcal{E},H} = \nabla^{\mathcal{E},H'} \circ \varepsilon_B.$$

Therefore ε_B induces an isomorphism (denoted by the same)

$$\varepsilon_B\colon H^\bullet(X,\mathcal{E},H) \to H^\bullet(X,\mathcal{E},H') \tag{1}$$

on the twisted de Rham cohomology. So the twisted Betti numbers depend only on the de Rham cohomology class of H. If they are finite, the Euler characteristic number

$$\chi(X,\mathcal{E},H) := \sum_{k=0,1} (-1)^k b_{\bar{k}}(X,\mathcal{E},H) = \chi(X,\mathcal{E}) = \chi(X)\,\mathrm{rk}\,\mathcal{E}$$

is independent of H and depends on $\mathcal{E}$ only through its rank. If X is odd-dimensional, then $\chi(X,\mathcal{E},H) = \chi(X,\mathcal{E}) = \chi(X) = 0$.

When H is a 1-form, $H^\bullet(X,\mathcal{E},H)$ has a $\mathbb{Z}$-grading but the dimension can jump when H is rescaled by a non-zero number [42, 43, 44]. The behavior is qualitatively different when the degree of H is at least 3.

Proposition 1.1. *Let $\mathcal{E}$ be a flat vector bundle over X and H, an odd-degree closed form on X. Suppose $H=\sum_{i\geq 1}H_{2i+1}$, where each H_{2i+1} is a $(2i+1)$-form. For any $\lambda\in\mathbb{R}$, let $H^{(\lambda)}=\sum_{i\geq 1}\lambda^{\frac{i-1}{2}}H_{2i+1}$. Then $H^\bullet(X,\mathcal{E},H)\cong H^\bullet(X,\mathcal{E},H^{(\lambda)})$ if $\lambda\neq 0$.*

Proof. For any λ, let c_λ act on $\Omega^\bullet(X,\mathcal{E})$ by multiplying $\lambda^{[\frac{i}{2}]}$ on i-forms. Then $H^{(\lambda)}=c_\lambda(H)$ and $c_\lambda\circ\nabla^{\mathcal{E},H}=\lambda^k\,\nabla^{\mathcal{E},H^{(\lambda)}}\circ c_\lambda$ on $\Omega^{\bar{k}}(X,\mathcal{E})$ for $k=0,1$. If $\lambda\neq 0$, then c_λ induces the desired isomorphism on twisted cohomology groups. □

Although the twisted differential $\nabla^{\mathcal{E},H}$ does not preserve the $\mathbb{Z}$-grading of the de Rham complex, it does respect a filtration F given by [50, 1]

$$F^p\Omega^{\bar{k}}(X,\mathcal{E})=\bigoplus_{\substack{i\geq p\\ i=k \bmod 2}}\Omega^i(X,\mathcal{E}).$$

This filtration gives rise to a spectral sequence $\{E_r^{pq},\delta_r\}$ converging to the twisted cohomology $H^\bullet(X,\mathcal{E},H)$. Without loss of generality, we assume that H contains no component of 1-form, which can be absorbed in the flat connection. That is, $H=H_3+H_5+\cdots$, where H_i is an i-form ($i=3,5,\cdots$). Then

$$E_2^{p\bar{q}}=\begin{cases}H^p(X,\mathcal{E}), & \text{if } q=0,\\ 0, & \text{if } q=1.\end{cases}$$

As usual, $E_{r+1}^\bullet$ is computed from a complex $(E_r^\bullet,\delta_r)$ for $r\geq 2$. We have $\delta_2=\delta_4=\cdots=0$, while $\delta_3,\delta_5,\cdots$ are given by the cup products with $[H_3],[H_5],\cdots$ and by the higher Massey products with them [50, 1]. Proposition 1.1 can also be derived by using this spectral sequence.

Until recently at least, there is no simplicial analogue of twisted de Rham cohomology except in a special situation in §5.2, as the cup product is not in general graded commutative at the level of cochain complex.

1.3. **Homotopy invariance of twisted de Rham cohomology.** Given X, $\mathcal{E}$ and H as above, any smooth map $f\colon Y\to X$ (where Y is another smooth manifold) induces a homomorphism

$$f^*\colon H^\bullet(X,\mathcal{E},H)\to H^\bullet(Y,f^*\mathcal{E},f^*H).$$

We will show that this map depends only on the homotopy class of f. For simplicity, we assume that $\mathcal{E}$ is a trivial line bundle with the trivial connection and denote $\nabla^{\mathcal{E},H}$ by d^H in this case. Let I be the unit interval.

Lemma 1.2. *Let $\pi\colon X\times I\to X$ denote the projection onto X and $s\colon X\to X\times I$, the map $x\mapsto(x,0)$ for $x\in X$. Then the maps $\pi^*\colon H^\bullet(X,H)\to H^\bullet(X\times I,\pi^*H)$ and $s^*\colon H^\bullet(X\times I,\pi^*H)\to H^\bullet(X,H)$ are inverses to each other.*

Proof. Clearly, $s^*\circ\pi^*$ is the identity map on $H^\bullet(X,H)$. By the homotopy invariance of de Rham cohomology (cf. §I.4 of [10]), there is a chain homotopy operator $K\colon\Omega^i(X\times I)\to\Omega^{i-1}(X)$ such that for any $\omega\in\Omega^\bullet(X\times I)$,

$$\omega-\pi^*s^*\omega=d\pi^*K\omega+\pi^*Kd\omega.$$

Since $K(\pi^*H\wedge\omega)=-H\wedge K(\omega)$, we have

$$\omega-\pi^*s^*\omega=d^H\pi^*K\omega+\pi^*Kd^{\pi^*H}\omega.$$

Therefore $\pi^*\circ s^*$ is the identity map on $H^\bullet(X\times I,\pi^*H)$. □

Proposition 1.3. *Let $f_0, f_1 \colon Y \to X$ be two smooth maps that are homotopic. Then there exists $B \in \Omega^{\bar{0}}(Y)$ such that $f_1^*H = f_0^*H - dB$ and the following diagram commutes*

$$\begin{tikzcd} & H^\bullet(X,H) \arrow[dl, "f_0^*"'] \arrow[dr, "f_1^*"] & \\ H^\bullet(Y, f_0^*H) \arrow[rr, "\varepsilon_B"] & & H^\bullet(Y, f_1^*H). \end{tikzcd}$$

Proof. Let $\pi\colon Y \times I \to Y$ be the projection onto Y. Define the smooth maps $s_j\colon Y \to Y \times I$ $(j = 0, 1)$ by $s_j(y) = (y, j)$, where $y \in Y$. Then a homotopy between f_0 and f_1 is a smooth map $F\colon Y \times I \to X$ such that $f_j = F \circ s_j$ for $j = 0, 1$. There exists $\tilde{B} \in \Omega^{\bar{0}}(Y \times I)$ such that $F^*H = \pi^* f_0^*H - d\tilde{B}$ and $s_0^*\tilde{B} = 0$. Let $B = s_1^*\tilde{B} \in \Omega^{\bar{0}}(Y)$ and $\tilde{B}' = \tilde{B} - \pi^*B$. Then $f_1^*H - f_0^*H = -dB$, $F^*H = \pi^* f_1^*H - d\tilde{B}'$ and $s_1^*\tilde{B}' = 0$. There is a commutative diagram

$$\begin{tikzcd} & & H^\bullet(Y \times I, f_0^*H) \arrow[dl, "\varepsilon_{\tilde{B}}"'] \arrow[dd, "\varepsilon_{\pi^*B}"] & H^\bullet(Y, f_0^*H) \arrow[l, "\pi^*"'] \arrow[dd, "\varepsilon_B"] \\ H^\bullet(X,H) \arrow[r, "F^*"] & H^\bullet(Y \times I, F^*H) & & \\ & & H^\bullet(Y \times I, f_1^*H) \arrow[ul, "\varepsilon_{\tilde{B}'}"] & H^\bullet(Y, f_1^*H). \arrow[l, "\pi^*"'] \end{tikzcd}$$

By Lemma 1.2, $s_0 = (\pi^*)^{-1}\colon H^\bullet(Y \times I, \pi^* f_0^*H) \to H^\bullet(Y, f_0^*H)$. Since $s_0^*\tilde{B} = 0$, $(\pi^*)^{-1} \circ \varepsilon_{\tilde{B}}^{-1} = s_0^*\colon H^\bullet(Y \times I, F^*H) \to H^\bullet(Y, f_0^*H)$. Similarly, $(\pi^*)^{-1} \circ \varepsilon_{\tilde{B}'}^{-1} = s_1^*\colon H^\bullet(Y \times I, F^*H) \to H^\bullet(Y, f_1^*H)$. The result follows since $f_j^* = s_j^* \circ F^*$, $j = 0, 1$. $\square$

It is clear from the proof that in addition to $f_1^*H = f_0^*H - dB$, B has to come from the homotopy. In fact, $B = -K(F^*H)$, where K is the homotopy chain map in the proof of Lemma 1.2. If H is fixed in the homotopy process, then $f_0^* = f_1^*$.

Corollary 1.4. *Suppose X, X' are smooth manifolds and H, H' are closed, odd-degree forms on X, X', respectively. If there is a smooth homotopy equivalence $f\colon X \to X'$ such that $[f^*H'] = [H]$, then $H^\bullet(X,H) \cong H^\bullet(X', H')$.*

2. Analytic torsion of twisted de Rham complexes

In this section, we define analytic torsion $\tau(X, \mathcal{E}, H) \in \det H^\bullet(X, \mathcal{E}, H)$ of the twisted de Rham complexes introduced in §1.2. Since these complexes are only $\mathbb{Z}_2$-graded, the twisted analytic torsion is more complicated to define and to study than its classical counterpart.

2.1. **The construction of analytic torsion.** To simplify notation, let $C^{\bar{k}} := \Omega^{\bar{k}}(X, \mathcal{E})$ and let $d_{\bar{k}} = d_{\bar{k}}^{\mathcal{E},H}$ be the operator $\nabla^{\mathcal{E},H}$ acting on $C^{\bar{k}}$ $(k = 0, 1)$. Then $d_{\bar{1}} d_{\bar{0}} = d_{\bar{0}} d_{\bar{1}} = 0$ and we have a complex

$$\cdots \xrightarrow{d_{\bar{1}}} C^{\bar{0}} \xrightarrow{d_{\bar{0}}} C^{\bar{1}} \xrightarrow{d_{\bar{1}}} C^{\bar{0}} \xrightarrow{d_{\bar{0}}} \cdots . \tag{2}$$

Denote by $d_{\bar{k}}^\dagger$ the adjoint of $d_{\bar{k}}$ with respect to the scalar product of §1.1. Then the Laplacians

$$\Delta_{\bar{k}} = \Delta_{\bar{k}}^{\mathcal{E},H} := d_{\bar{k}}^\dagger d_{\bar{k}} + d_{\overline{k+1}} d_{\overline{k+1}}^\dagger, \quad k = 0, 1$$

are elliptic operators and therefore the complex (2) is elliptic. By Hodge theory, the natural map $\ker(\Delta_{\bar{k}}) \to H^{\bar{k}}(X, \mathcal{E}, H)$ taking each twisted harmonic form to

its cohomology class is an isomorphism. Ellipticity of $\Delta_{\bar k}$ ensures that the twisted Betti numbers $b_{\bar k}(X,\mathcal{E},H)$ ($k=0,1$) are finite.

The scalar product on $C^{\bar k}$ restricts to one on the space of twisted harmonic forms $\ker(\Delta_{\bar k})\cong H^{\bar k}(X,\mathcal{E},H)$. Let $\{\nu_{\bar k,i}\}_{i=1}^{b_{\bar k}}$ be an oriented orthonormal basis of $H^{\bar k}(X,\mathcal{E},H)$ and let $\eta_{\bar k}=\eta_{\bar k}^{\mathcal{E},H}:=\nu_{\bar k,1}\wedge\cdots\wedge\nu_{\bar k,b_{\bar k}}$, the unit volume element. Then $\eta_{\bar 0}\otimes\eta_{\bar 1}^{-1}\in\det H^\bullet(X,\mathcal{E},H)$. The *analytic torsion of the twisted de Rham complex* is defined to be

$$\tau(X,\mathcal{E},H):=(\mathrm{Det}'\,d_{\bar 0}^\dagger d_{\bar 0})^{1/2}(\mathrm{Det}'\,d_{\bar 1}^\dagger d_{\bar 1})^{-1/2}\eta_{\bar 0}\otimes\eta_{\bar 1}^{-1}\in\det H^\bullet(X,\mathcal{E},H), \tag{3}$$

where $\mathrm{Det}'\,d_{\bar k}^\dagger d_{\bar k}$ denotes the zeta-function regularized determinant of $d_{\bar k}^\dagger d_{\bar k}$ on the orthogonal complement of its kernel. The next subsection is devoted to showing that these determinants make sense. When $\mathcal{E}$ is the trivial line bundle over X with the trivial connection, we set $\tau(X,H)=\tau(X,\mathcal{E},H)$.

We explain the motivation for definition (3) by considering the case $H=0$. Then $C^{\bar k}=\bigoplus_{i=k\bmod 2}C^i$ and $d_{\bar k}=\sum_{i=k\bmod 2}d_i$ ($k=0,1$), where $C^i=\Omega^i(X,\mathcal{E})$ ($0\le i\le n$) and the differentials $d_i=d_i^{\mathcal{E}}$ ($0\le i\le n-1$) form the $\mathbb{Z}$-graded de Rham complex

$$0\to C^0\xrightarrow{d_0}C^1\xrightarrow{d_1}\cdots\xrightarrow{d_{n-1}}C^n\to 0$$

with $d_id_{i-1}=0$ ($1\le i\le n$). By spectral theory, $\mathrm{Det}'\,d_i^\dagger d_i$ ($0\le i\le n-1$) can be defined and is equal to $\prod_{j=0}^{n-i-1}(\mathrm{Det}'\,\Delta_i)^{(-1)^j}$, where $\Delta_i=d_i^\dagger d_i+d_{i-1}d_{i-1}^\dagger$ (with $d_{-1}=d_n=0$) is the Laplacian on C^i. Thus the determinant factor in (3) is

$$(\mathrm{Det}'\,d_{\bar 0}^\dagger d_{\bar 0})^{1/2}(\mathrm{Det}'\,d_{\bar 1}^\dagger d_{\bar 1})^{-1/2}=\prod_{i=0}^{n-1}(\mathrm{Det}'\,d_i^\dagger d_i)^{(-1)^i/2}=\prod_{i=0}^{n}(\mathrm{Det}'\,\Delta_i)^{(-1)^{i+1}i/2},$$

which yields the Ray-Singer torsion $\tau(X,\mathcal{E})$ [47]. When $\mathcal{E}$ is the trivial line bundle over X, we set $\tau(X)=\tau(X,\mathcal{E})$.

We wish to point out that the classical signature or Dirac complex, although being a 2-term complex, is not of the form (2) because it does not satisfy $d_{\bar 1}d_{\bar 0}=d_{\bar 0}d_{\bar 1}=0$. Therefore, no torsion is defined in these cases.

If $\mathcal{E}$ is a complex vector bundle, the torsion is only defined up to a phase due to the ambiguity in the choice of the unit volume elements $\eta_{\bar k}$. Therefore an equality of torsions means that they are equal up to a phase or that the volume elements can be chosen so that they are equal. More intrinsic is the norm on the determinant line (cf. [45, 3, 8] for the $\mathbb{Z}$-graded case) given by

$$||\cdot||=(\mathrm{Det}'\,d_{\bar 0}^\dagger d_{\bar 0})^{1/2}(\mathrm{Det}'\,d_{\bar 1}^\dagger d_{\bar 1})^{-1/2}\,|\cdot|,$$

where $|\cdot|$ is the norm induced by the scalar products on $\ker(\Delta_{\bar k})\cong H^{\bar k}(X,\mathcal{E},H)$, $k=0,1$. However, to facilitate the presentation, we will still regard torsions as (equivalent classes of) elements in the determinant lines. Recently, refined and complex-valued analytic torsions were introduced as well-defined elements of the determinant line [13, 14, 18].

2.2. **The zeta-function regularized determinants.** Given a semi-positive definite self-adjoint operator A, the *zeta-function* of A (whenever it is defined) is

$$\zeta(s,A):=\mathrm{Tr}'\,A^{-s},$$

where Tr' stands for the trace restricted to the subspace orthogonal to $\ker(A)$. If $\zeta(s,A)$ can be extended meromorphically in s so that it is holomorphic at $s=0$, then the *zeta-function regularized determinant* of A is defined as

$$\mathrm{Det}'\,A = e^{-\zeta'(0,A)}.$$

If A is an elliptic differential operator of order m on a compact manifold of dimension n, then $\zeta(s,A)$ is holomorphic when $\Re(s) > n/m$ and can be extended meromorphically to the entire complex plane with possible simple poles at $\{\frac{n-l}{m}, l = 0,1,2,\dots\}$ only [52] (cf. [53]). Moreover, the extended function is holomorphic at $s=0$ and therefore the determinant $\mathrm{Det}'\,A$ is defined for such an operator. Examples are the Laplacians $\Delta_i^{\mathcal{E}}$ acting on i-forms on a compact Riemannian manifold X with values in a vector bundle $\mathcal{E}$ with an Hermitian structure; their determinants $\mathrm{Det}'\,\Delta_i^{\mathcal{E}}$ enter the Ray-Singer analytic torsion for the de Rham complex [47, 48]. For the twisted de Rham complex (2), the Laplacians $\Delta_{\bar{k}} = \Delta_{\bar{k}}^{\mathcal{E},H}$ $(k=0,1)$ acting on even/odd-degree forms are also elliptic, and therefore the determinants $\mathrm{Det}'\,\Delta_{\bar{k}}$ $(k=0,1)$ still make sense (and are in fact equal). However, what appear in the twisted analytic torsion (3) are not these determinants, but $\mathrm{Det}'\,d_{\bar{k}}^\dagger d_{\bar{k}}$, which are much harder to define.

Let $\mathrm{spec}(A)$ ($\mathrm{spec}'(A)$, respectively) be the set of eigenvalues (positive eigenvalues, respectively) of A. For any $\lambda\in\mathrm{spec}(A)$, let $m(\lambda,A)$ be its multiplicity (if it is finite). Then

$$\zeta(s,A) = \sum_{\lambda\in\mathrm{spec}'(A)} \frac{m(\lambda,A)}{\lambda^s}.$$

Given a flat vector bundle $\mathcal{E}$ over a manifold X and a closed odd-degree form H on X, set $\mathrm{spec}_{\mathrm{I}}(\Delta_{\bar{0}}) := \mathrm{spec}(\Delta_{\bar{0}}|_{\mathrm{im}(d_{\bar{0}}^\dagger)}) = \mathrm{spec}'(d_{\bar{0}}^\dagger d_{\bar{0}})$, $m_{\mathrm{I}}(\lambda,\Delta_{\bar{0}}) := m(\lambda,\Delta_{\bar{0}}|_{\mathrm{im}(d_{\bar{0}}^\dagger)})$ and $\mathrm{spec}_{\mathrm{II}}(\Delta_{\bar{0}}) := \mathrm{spec}(\Delta_{\bar{0}}|_{\mathrm{im}(d_{\bar{1}})}) = \mathrm{spec}'(d_{\bar{1}}^\dagger d_{\bar{1}})$, $m_{\mathrm{II}}(\lambda,\Delta_{\bar{0}}) := m(\lambda,\Delta_{\bar{0}}|_{\mathrm{im}(d_{\bar{1}})})$. Since $\Delta_{\bar{0}}$ is diagonal with respect to the decomposition $C^{\bar{0}} = \mathrm{im}(d_{\bar{0}}^\dagger)\oplus\mathrm{im}(d_{\bar{1}})\oplus\ker(\Delta_{\bar{0}})$, we have $\mathrm{spec}'(\Delta_{\bar{0}}) = \mathrm{spec}_{\mathrm{I}}(\Delta_{\bar{0}})\cup\mathrm{spec}_{\mathrm{II}}(\Delta_{\bar{0}})$ and $m(\lambda,\Delta_{\bar{0}}) = m_{\mathrm{I}}(\lambda,\Delta_{\bar{0}}) + m_{\mathrm{II}}(\lambda,\Delta_{\bar{0}})$ if $\lambda>0$. Therefore

$$\zeta(s,d_{\bar{0}}^\dagger d_{\bar{0}}) = \sum_{\lambda\in\mathrm{spec}_{\mathrm{I}}(\Delta_{\bar{0}})} \frac{m_{\mathrm{I}}(\lambda,\Delta_{\bar{0}})}{\lambda^s}, \qquad \zeta(s,d_{\bar{1}}^\dagger d_{\bar{1}}) = \sum_{\lambda\in\mathrm{spec}_{\mathrm{II}}(\Delta_{\bar{0}})} \frac{m_{\mathrm{II}}(\lambda,\Delta_{\bar{0}})}{\lambda^s}. \tag{4}$$

The sum of the two zeta-functions is

$$\sum_{k=0,1} \zeta(s,d_{\bar{k}}^\dagger d_{\bar{k}}) = \zeta(s,\Delta_{\bar{0}}) = \zeta(s,\Delta_{\bar{1}}). \tag{5}$$

However, what we need for (3) is their difference.

Theorem 2.1. *For $k=0,1$, $\zeta(s,d_{\bar{k}}^\dagger d_{\bar{k}})$ is holomorphic in the half plane for $\Re(s) > n/2$ and extends meromorphically to $\mathbb{C}$ with possible simple poles at $\{\frac{n-l}{2}, l = 0,1,2,\dots\}$ only, and is holomorphic at $s=0$.*

Proof. Let $P_{\bar{k}}$ $(k=0,1)$ be the orthogonal projection onto the closure of the subspace $\mathrm{im}(d_{\bar{k}}^\dagger)$. As $d_{\bar{k}}d_{\bar{k}}^\dagger$ and $\Delta_{\bar{k}}$ are equal and invertible on (the closure of) $\mathrm{im}(d_{\bar{k}})$, we have

$$P_{\bar{k}} = d_{\bar{k}}^\dagger(d_{\bar{k}}d_{\bar{k}}^\dagger)^{-1}d_{\bar{k}} = d_{\bar{k}}^\dagger(\Delta_{\bar{k}})^{-1}d_{\bar{k}},$$

which implies that $P_{\bar{k}}$ is a pseudodifferential operator of order 0. Moreover,

$$\zeta(s,d_{\bar{k}}^\dagger d_{\bar{k}}) = \mathrm{Tr}(P_{\bar{k}}\Delta_{\bar{k}}^{-s}).$$

By general theory [27], $\zeta(s, d_{\bar{k}}^{\dagger} d_{\bar{k}})$ is holomorphic in the half plane $\Re(s) > n/2$ and extends meromorphically to $\mathbb{C}$ with possible simple poles at $\{\frac{n-l}{2}, l = 0, 1, 2, \dots\}$ only. The Laurent series of $\zeta(s, d_{\bar{k}}^{\dagger} d_{\bar{k}})$ at $s = 0$ is

$$\mathrm{Tr}(P_{\bar{k}} \Delta_{\bar{k}}^{-s}) = \frac{c_{-1}(P_{\bar{k}}, \Delta_{\bar{k}})}{s} + c_0(P_{\bar{k}}, \Delta_{\bar{k}}) + \sum_{l=1}^{\infty} c_l(P_{\bar{k}}, \Delta_{\bar{k}})\, s^l.$$

Here $c_{-1}(P_{\bar{k}}, \Delta_{\bar{k}}) = \frac{1}{2} \operatorname{res}(P_{\bar{k}})$, where $\operatorname{res}(P_{\bar{k}})$ is known as the non-commutative residue or the Guillemin-Wodzicki residue trace of $P_{\bar{k}}$ [56, 30]. Since $P_{\bar{k}}$ is a projection, $\operatorname{res}(P_{\bar{k}}) = 0$ [56, 15, 25]. Therefore $\zeta(s, d_{\bar{k}}^{\dagger} d_{\bar{k}})$ is regular at $s = 0$. □

Theorem 2.1 justifies the definition of the twisted analytic torsion in (3). The constant term $\zeta(0, d_{\bar{k}}^{\dagger} d_{\bar{k}}) = c_0(P_{\bar{k}}, \Delta_{\bar{k}})$ of the above Laurent series is related to the Kontsevich-Vishik trace [33, 26]. It can nevertheless be studied by standard heat kernel techniques (Lemma 2.2, Corollaries 3.2 and 3.6 below). Recall that the zeta-function is related to the heat kernel by a Mellin transform

$$\zeta(s, A) = \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} \,\mathrm{Tr}'\, e^{-tA}\, dt.$$

Lemma 2.2. *If* $\dim X$ *is odd, then*

$$\sum_{k=0,1} \zeta(0, d_{\bar{k}}^{\dagger} d_{\bar{k}}) = -b_{\bar{0}}(X, \mathcal{E}, H) = -b_{\bar{1}}(X, \mathcal{E}, H).$$

Proof. When $n = \dim X$ is odd, $b_{\bar{0}}(X, \mathcal{E}, H) = b_{\bar{1}}(X, \mathcal{E}, H)$ as $\chi(X, \mathcal{E}, H) = 0$. By (5), it suffices to show that $\zeta(0, \Delta_{\bar{k}}) = -b_{\bar{k}}(X, \mathcal{E}, H)$, $k = 0, 1$. By the asymptotic expansion of the heat kernel (cf. [24, 2]),

$$\mathrm{Tr}\, e^{-t\Delta_{\bar{k}}} \sim \sum_{l=0}^{\infty} c_{\bar{k},l}\, t^{-n/2+l}$$

as $t \downarrow 0$, where $c_{\bar{k},l} \in \mathbb{R}$. We have, for $\Re(s) > n/2$,

$$\begin{aligned} \zeta(s, \Delta_{\bar{k}}) &= \frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} (\mathrm{Tr}\, e^{-t\Delta_{\bar{k}}} - b_{\bar{k}})\, dt \\ &= \frac{1}{\Gamma(s)} \Big(-\frac{b_{\bar{k}}}{s} + \sum_{l=0}^{N} \frac{c_{\bar{k},l}}{s - n/2 + l} + R_N(s) \Big), \end{aligned}$$

where N is a sufficiently large integer. Here $R_N(s)$ is holomorphic when $\Re(s) > n/2 - N$. Since n is odd and since $\Gamma(s)$ has a simple pole at $s = 0$, the result follows. □

3. Twisted analytic torsion under metric and flux deformations

3.1. **Variation of analytic torsion with respect to the metrics.** We assume that X is a closed compact oriented manifold of odd dimension. Let g_X be a Riemannian metric on X and $g_{\mathcal{E}}$, an Hermitian metric on $\mathcal{E}$. Let $Q_{\bar{k}}$ ($k = 0, 1$) be the orthogonal projection from (the completion of) $C^{\bar{k}}$ to $\ker(\Delta_{\bar{k}})$. Suppose the pair $(g_X, g_{\mathcal{E}})$ is deformed smoothly along a one-parameter family with parameter $u \in \mathbb{R}$, then the operators $*$, $\sharp$ and Γ all depend smoothly on u. Let

$$\alpha = \Gamma^{-1} \frac{\partial \Gamma}{\partial u}.$$

We show the invariance of the analytic torsion (3) by showing in the next two lemmas that the variation of the regularized determinants cancels that of the volume elements.

Lemma 3.1. *Under the above assumptions,*

$$\frac{\partial}{\partial u}\log[\mathrm{Det}'\, d_{\bar 0}^\dagger d_{\bar 0}\,(\mathrm{Det}'\, d_{\bar 1}^\dagger d_{\bar 1})^{-1}] = \sum_{k=0,1}(-1)^k\,\mathrm{Tr}(\alpha Q_{\bar k}).$$

Proof. While $d_{\bar k}$ in independent of u, we have

$$\frac{\partial d_{\bar k}^\dagger}{\partial u} = -[\alpha, d_{\bar k}^\dagger],$$

which follows easily from $d_{\bar k}^\dagger = \pm\Gamma^{-1}\circ d_{\bar k}^{\mathcal{E}^*,H}\circ\Gamma$ (since $n=\dim X$ is odd). Using $P_{\bar k}d_{\bar k}^\dagger = d_{\bar k}^\dagger$, $d_{\bar k}P_{\bar k} = d_{\bar k}$ and $P_{\bar k}^2 = P_{\bar k}$, we get $d_{\bar k}^\dagger d_{\bar k}P_{\bar k} = P_{\bar k}d_{\bar k}^\dagger d_{\bar k} = d_{\bar k}^\dagger d_{\bar k}$ and

$$\frac{\partial P_{\bar k}}{\partial u} = \frac{\partial P_{\bar k}}{\partial u}P_{\bar k}, \qquad P_{\bar k}\frac{\partial P_{\bar k}}{\partial u} = 0.$$

Following the $\mathbb{Z}$-graded case [47, 48], we set

$$\begin{aligned} f(s,u) &= \sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\,\mathrm{Tr}(e^{-td_{\bar k}^\dagger d_{\bar k}}P_{\bar k})\,dt \\ &= \Gamma(s)\sum_{k=0,1}(-1)^k\zeta(s, d_{\bar k}^\dagger d_{\bar k}). \end{aligned}$$

Using the above identities on $P_{\bar k}$, the trace property and by an application of Duhamel's principle, we get

$$\begin{aligned} \frac{\partial f}{\partial u} &= \sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\,\mathrm{Tr}\left(t[\alpha, d_{\bar k}^\dagger]d_{\bar k}e^{-td_{\bar k}^\dagger d_{\bar k}} + e^{-td_{\bar k}^\dagger d_{\bar k}}\frac{\partial P_{\bar k}}{\partial u}P_{\bar k}\right)dt \\ &= \sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\,\mathrm{Tr}\left(t\alpha[d_{\bar k}^\dagger, d_{\bar k}e^{-td_{\bar k}^\dagger d_{\bar k}}] + P_{\bar k}e^{-td_{\bar k}^\dagger d_{\bar k}}\frac{\partial P_{\bar k}}{\partial u}\right)dt \\ &= \sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\,\mathrm{Tr}\left(t\alpha(e^{-td_{\bar k}^\dagger d_{\bar k}}d_{\bar k}^\dagger d_{\bar k} - e^{-td_{\bar k}d_{\bar k}^\dagger}d_{\bar k}d_{\bar k}^\dagger) + e^{-td_{\bar k}^\dagger d_{\bar k}}P_{\bar k}\frac{\partial P_{\bar k}}{\partial u}\right)dt \\ &= \sum_{k=0,1}(-1)^k\int_0^\infty t^s\,\mathrm{Tr}(\alpha e^{-t\Delta_{\bar k}}\Delta_{\bar k})\,dt \\ &= -\sum_{k=0,1}(-1)^k\int_0^\infty t^s\frac{\partial}{\partial t}\,\mathrm{Tr}\left(\alpha(e^{-t\Delta_{\bar k}} - Q_{\bar k})\right)dt. \end{aligned}$$

Integrating by parts, we have

$$\begin{aligned} \frac{\partial f}{\partial u} &= s\sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\,\mathrm{Tr}\left(\alpha(e^{-t\Delta_{\bar k}} - Q_{\bar k})\right)dt \\ &= s\sum_{k=0,1}(-1)^k\left(\int_0^1 + \int_1^\infty\right)t^{s-1}\,\mathrm{Tr}\left(\alpha(e^{-t\Delta_{\bar k}} - Q_{\bar k})\right)dt. \end{aligned}$$

Since α is a smooth tensor and n is odd, the asymptotic expansion as $t\downarrow 0$ for $\mathrm{Tr}(\alpha e^{-t\Delta_{\bar k}})$ does not contain a constant term. Therefore $\int_0^1 t^{s-1}\,\mathrm{Tr}(\alpha e^{-t\Delta_{\bar k}})\,dt$ does not have a pole at $s=0$. On the other hand, because of the exponential decay

of $\mathrm{Tr}(\alpha(e^{-t\Delta_{\bar k}} - Q_{\bar k}))$ for large t, $\int_1^\infty t^{s-1}\,\mathrm{Tr}(\alpha(e^{-t\Delta_{\bar k}} - Q_{\bar k}))\,dt$ is an entire function in s. So

$$\begin{aligned}\frac{\partial f}{\partial u}\Big|_{s=0} &= -s\sum_{k=0,1}(-1)^k\int_0^1 t^{s-1}\,\mathrm{Tr}(\alpha Q_{\bar k})\,dt\Big|_{s=0}\\ &= -\sum_{k=0,1}(-1)^k\,\mathrm{Tr}(\alpha Q_{\bar k}) \qquad (6)\end{aligned}$$

and hence

$$\frac{\partial}{\partial u}\sum_{k=0,1}(-1)^k\zeta(0, d_{\bar k}^\dagger d_{\bar k}) = 0. \qquad (7)$$

Finally, the result follows from (6), (7) and

$$\log[\mathrm{Det}'\, d_{\bar 0}^\dagger d_{\bar 0}\,(\mathrm{Det}'\, d_{\bar 1}^\dagger d_{\bar 1})^{-1}] = -\lim_{s\to 0}\Big[f(s,u) - \frac{1}{s}\sum_{k=0,1}(-1)^k\zeta(0, d_{\bar k}^\dagger d_{\bar k})\Big].$$

□

Corollary 3.2. *Under the above deformation, each* $\zeta(0, d_{\bar k}^\dagger d_{\bar k})$ *(*$k = 0, 1$*) is invariant.*

Proof. By (7), their difference is invariant. By Lemma 2.2, their sum is also invariant since $b_{\bar 0}(X, \mathcal{E}, H)$ is defined without using the metrics. □

Lemma 3.3. *Under the same assumptions, along any one-parameter deformation of* $(g_X, g_{\mathcal{E}})$*, the volume elements* $\eta_{\bar 0}$*,* $\eta_{\bar 1}$ *can be chosen so that*

$$\frac{\partial}{\partial u}(\eta_{\bar 0}\otimes\eta_{\bar 1}^{-1}) = -\frac{1}{2}\sum_{k=0,1}(-1)^k\,\mathrm{Tr}(\alpha Q_{\bar k})\,\eta_{\bar 0}\otimes\eta_{\bar 1}^{-1}.$$

Proof. Recall that $\eta_{\bar k} = \nu_{\bar k,1}\wedge\cdots\wedge\nu_{\bar k,b_{\bar k}}$, where $\{\nu_{\bar k,i}\}_{i=1}^{b_{\bar k}}$ is an orthonormal basis of $H^{\bar k}(X, \mathcal{E}, H)$, $k = 0, 1$. Since $(\nu_{\bar k,i}, \nu_{\bar k,j}) = \int_X \nu_{\bar k,i}\wedge\Gamma\,\nu_{\bar k,j} = \delta_{ij}$, we get, by taking the derivative with respect to u,

$$\Re\left(\frac{\partial\nu_{\bar k,i}}{\partial u}, \nu_{\bar k,i}\right) = -\frac{1}{2}(\nu_{\bar k,i}, \alpha\nu_{\bar k,i}).$$

We can adjust the phase of $\nu_{\bar k,i}$ so that $(\frac{\partial\nu_{\bar k,i}}{\partial u}, \nu_{\bar k,i})$ is real. Since we identify $\det\ker(\Delta_{\bar k})$ with $\det H^{\bar k}(X, \mathcal{E}, H)$ along the deformation, we have

$$\begin{aligned}\frac{\partial\eta_{\bar k}}{\partial u} &= \sum_{i=1}^{b_{\bar k}}\nu_{\bar k,1}\wedge\cdots\wedge\frac{\partial\nu_{\bar k,i}}{\partial u}\wedge\cdots\wedge\nu_{\bar k,b_{\bar k}}\\ &= -\frac{1}{2}\sum_{i=1}^{b_{\bar k}}(\nu_{\bar k,i}, \alpha\,\nu_{\bar k,i})\,\eta_{\bar k}\\ &= -\frac{1}{2}\,\mathrm{Tr}(\alpha Q_{\bar k})\,\eta_{\bar k}.\end{aligned}$$

The result follows. □

Combining Lemma 3.1 and Lemma 3.3, we have

Theorem 3.4 (metric independence of analytic torsion)**.** *Let X be a closed, compact manifold of odd dimension, $\mathcal{E}$, a flat vector bundle over X, and H, a closed differential form on X of odd degree. Then the analytic torsion $\tau(X,\mathcal{E},H)$ of the twisted de Rham complex does not depend on the choice of the Riemannian metric on X or the Hermitian metric on $\mathcal{E}$.*

3.2. **Variation of analytic torsion with respect to the flux in a cohomology class.** We continue to assume that $\dim X$ is odd and use the same notation as above. Suppose the (real) flux form H is deformed smoothly along a one-parameter family with parameter $v\in\mathbb{R}$ in such a way that the cohomology class $[H]\in H^{\bar 1}(X,\mathbb{R})$ is fixed. Then $\frac{\partial H}{\partial v}=-dB$ for some form $B\in\Omega^{\bar 0}(X)$ that depends smoothly on v; let

$$\beta=B\wedge\cdot\,.$$

As before, we show in the next two lemmas that the variation of the regularized determinants cancels that of the volume elements.

Lemma 3.5. *Under the above assumptions,*

$$\frac{\partial}{\partial v}\log[\mathrm{Det}'\,d_{\bar 0}^\dagger d_{\bar 0}\,(\mathrm{Det}'\,d_{\bar 1}^\dagger d_{\bar 1})^{-1}]=2\sum_{k=0,1}(-1)^k\,\mathrm{Tr}(\beta Q_{\bar k}).$$

Proof. As in the proof of Lemma 3.1, we set

$$f(s,v)=\sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\,\mathrm{Tr}(e^{-td_{\bar k}^\dagger d_{\bar k}}P_{\bar k})\,dt.$$

We note that B, hence β is real. Using

$$\frac{\partial d_{\bar k}}{\partial v}=[\beta,d_{\bar k}],\qquad \frac{\partial d_{\bar k}^\dagger}{\partial v}=-[\beta^\dagger,d_{\bar k}^\dagger],$$

$$P_{\bar k}^2=P_{\bar k}=P_{\bar k}^\dagger,\qquad P_{\bar k}\frac{\partial P_{\bar k}}{\partial v}P_{\bar k}=0$$

and by Dumahel's principle, we get

$$\begin{aligned}\frac{\partial f}{\partial v}&=\sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\,\mathrm{Tr}\left(t([\beta^\dagger,d_{\bar k}^\dagger]d_{\bar k}-d_{\bar k}^\dagger[\beta,d_{\bar k}])e^{-td_{\bar k}^\dagger d_{\bar k}}+e^{-td_{\bar k}^\dagger d_{\bar k}}\{\tfrac{\partial P_{\bar k}}{\partial u},P_{\bar k}\}\right)dt\\&=2\sum_{k=0,1}(-1)^k\int_0^\infty t^{s-1}\,\mathrm{Tr}\left(t\beta(e^{-td_{\bar k}^\dagger d_{\bar k}}d_{\bar k}^\dagger d_{\bar k}-e^{-td_{\bar k}d_{\bar k}^\dagger}d_{\bar k}d_{\bar k}^\dagger)+e^{-td_{\bar k}^\dagger d_{\bar k}}P_{\bar k}\tfrac{\partial P_{\bar k}}{\partial u}P_{\bar k}\right)dt\\&=2\sum_{k=0,1}(-1)^k\int_0^\infty t^s\,\mathrm{Tr}(\beta e^{-t\Delta_{\bar k}}\Delta_{\bar k})\,dt\\&=-2\sum_{k=0,1}(-1)^k\int_0^\infty t^s\frac{\partial}{\partial t}\,\mathrm{Tr}\left(\beta(e^{-t\Delta_{\bar k}}-Q_{\bar k})\right)dt.\end{aligned}$$

The rest is similar to the proof of Lemma 3.1. ☐

Corollary 3.6. *Under the above deformation, each $\zeta(0,d_{\bar k}^\dagger d_{\bar k})$ ($k=0,1$) is invariant.*

Proof. We follow the proof of Corollary 3.2, using the fact that $b_{\bar 0}(X,\mathcal{E},H)$ depends only on the cohomology class of H. ☐

If n is odd and $H=0$, then $\zeta(0,\Delta_i^{\mathcal{E}})=-b_i(X,\mathcal{E})$ and

$$\zeta(0,d_{\bar 0}^\dagger d_{\bar 0})=\sum_{i=1}^{(n+1)/2} i\,(b_{2i}(X,\mathcal{E})-b_{2i-1}(X,\mathcal{E})),$$

$$\zeta(0,d_{\bar 1}^\dagger d_{\bar 1})=\sum_{i=1}^{(n-1)/2} i\,(b_{2i+1}(X,\mathcal{E})-b_{2i}(X,\mathcal{E})).$$

We hope that when n is odd but $H\neq 0$, $\zeta(0,d_{\bar k}^\dagger d_{\bar k})$ can also be expressed in terms of topological numbers that are invariant under homotopy equivalences preserving $[H]$.

Lemma 3.7. *Under the same assumptions, along any one-parameter deformation of H that fixes the cohomology class $[H]$, the volume elements $\eta_{\bar 0}$, $\eta_{\bar 1}$ can be chosen so that*

$$\frac{\partial}{\partial v}(\eta_{\bar 0}\otimes\eta_{\bar 1}^{-1})=-\sum_{k=0,1}(-1)^k\operatorname{Tr}(\beta Q_{\bar k})\,\eta_{\bar 0}\otimes\eta_{\bar 1}^{-1},$$

where we identify $\det H^\bullet(X,\mathcal{E},H)$ *along the deformation using (1).*

Proof. Fix a reference point, say $v=0$, and let $H^{(0)}$, $\eta_{\bar k}^{(0)}$ be the values of H, $\eta_{\bar k}$, respectively, at $v=0$. To compare the volume elements $\eta_{\bar k}\in\det H^{\bar k}(X,\mathcal{E},H)$ at different values of v, we map them to $\det H^{\bar k}(X,\mathcal{E},H^{(0)})$ by the inverse of the isomorphism

$$\det\varepsilon_B\colon\ \det H^\bullet(X,\mathcal{E},H^{(0)})\to\det H^\bullet(X,\mathcal{E},H)$$

induced by (1). Since $\varepsilon_B=e^\beta$ on $\Omega^\bullet(X,\mathcal{E})$, we have, for $k=0,1$,

$$\frac{\partial}{\partial v}(\det\varepsilon_B)^{-1}\eta_{\bar k}=-\operatorname{Tr}(\beta Q_{\bar k})\,(\det\varepsilon_B)^{-1}\eta_{\bar k}.$$

The result follows. □

Combining Lemma 3.5 and Lemma 3.7, we have

Theorem 3.8 (flux representative independence of analytic torsion)**.** *Let X be a closed, compact manifold of odd dimension, $\mathcal{E}$, a flat vector bundle over X. Suppose H and H' are closed differential forms on X of odd degrees representing the same de Rham cohomology class, and let B be an even form so that $H'=H-dB$. Then the analytic torsion $\tau(X,\mathcal{E},H')=(\det\varepsilon_B)(\tau(X,\mathcal{E},H))$.*

3.3. **Relation to generalized geometry.** We specialize to the interesting case when H is a 3-form and explain the relation to generalized geometry [31, 28]. Recall that the bundle $TX\oplus T^*X$ has a bilinear form of signature (n,n) given by

$$\langle\xi_1+W_1,\xi_2+W_2\rangle:=(\xi_1(W_2)+\xi_2(W_1))/2,$$

where for $i=1,2$, ξ_i are 1-forms and W_i are vector fields on X. A *generalized metric* on X is a reduction of the structure group $O(n,n)$ to $O(n)\times O(n)$. Equivalently, a generalized metric is a splitting of $TX\oplus T^*X$ to a direct sum of two sub-bundles of rank n so that the bilinear form is positive on one and negative on the other. The positive sub-bundle is the graph of $g+B\in\Gamma(\operatorname{Hom}(TX,T^*X))$, where $g=g_X$ is a usual Riemannian metric on X and B is a 2-form on X.

A generalized metric on X defines as follows a scalar product, called the Born-Infeld metric [29] on $\Omega^\bullet(X)$. Let σ be the isomorphism from $\Omega^\bullet(X)$ to itself so that

if ω is the wedge product of 1-forms, then $\sigma(\omega)$ is the product with the order of 1-forms reversed. Thus $\sigma(\omega\wedge\omega')=\sigma(\omega')\wedge\sigma(\omega)$ for any forms ω, ω' and $\sigma(B)=-B$ if B is a 2-form. Choose a (local) orthonormal frame $\{e_i, i=1,\dots,n\}$ on X with respect to g and let $\hat{e}_i:=\iota_{e_i}+\iota_{e_i}(g+B)\wedge\cdot$ $(i=1,\dots,n)$ be operators acting on forms. Define a new star operation by

$$\star_{(g,B)}\,\omega=\sigma(\hat{e}_n\cdots\hat{e}_2\hat{e}_1\omega).$$

When $B=0$, $\star_{(g,B)}$ is the usual Hodge star $*_g$ given by g. The *Born-Infeld metric* (scalar product) on $\Omega^\bullet(X)^{\mathbb{C}}$ is [29]

$$(\omega',\omega)_{(g,B)}:=\int_X\overline{\omega'}\wedge\star_{(g,B)}\,\omega$$

for $\omega,\omega'\in\Omega^\bullet(X)^{\mathbb{C}}$. We show that the isomorphism ε_B intertwines the Born-Infeld metric $(\cdot,\cdot)_{(g,B)}$ and the usual scalar product $(\cdot,\cdot)_g$ defined by the Riemannian metric g.

Proposition 3.9. *For any $\omega,\omega'\in\Omega^\bullet(X)^{\mathbb{C}}$, we have*

$$(\omega',\omega)_{(g,B)}=(\varepsilon_B(\omega'),\varepsilon_B(\omega))_g.$$

Proof. Since

$$(\varepsilon_B^{-1}(\omega'),\varepsilon_B^{-1}(\omega))_{(g,B)}=\int_X\overline{\omega'}\wedge\sigma(\varepsilon_B\,\hat{e}_n\cdots\hat{e}_1\,\varepsilon_B^{-1}\omega),$$

it suffices to check that $\varepsilon_B\,\hat{e}_n\cdots\hat{e}_1\,\varepsilon_B^{-1}$ is independent of B. We replace B by vB, where $v\in\mathbb{R}$. Then, since $\frac{\partial\hat{e}_i}{\partial v}=\iota_{e_i}B\wedge\cdot=-[\beta,\hat{e}_i]$, we get

$$\frac{\partial}{\partial v}\big(\varepsilon_{vB}\,\hat{e}_n\cdots\hat{e}_1\,\varepsilon_{vB}^{-1}\big)=\varepsilon_{vB}\Big([\beta,\hat{e}_n\cdots\hat{e}_1]+\sum_{i=1}^n\hat{e}_n\cdots\frac{\partial\hat{e}_i}{\partial v}\cdots\hat{e}_1\Big)\varepsilon_{vB}^{-1}=0$$

and the result follows. ☐

Consider the variation $H\mapsto H'=H-dB$, where B is a 2-form known as the B-field. For simplicity, we take $\mathcal{E}$ as the trivial line bundle over X. As before, the operator $d^{H'}$ is conjugate to d^H by the isomorphism $\varepsilon_B=e^\beta$. As a consequence of Proposition 3.9, the adjoint (defined in [29]) of d^H with respect to the Born-Infeld metric, and hence the corresponding Laplacian, are also conjugate to $(d^{H'})^\dagger$ and $\Delta^{H'}$, respectively, via the same isomorphism ε_B.

We thus conclude that deformation of H by a B-field is equivalent to deforming the usual metric to a generalized metric. Theorem 3.8 states that the torsion is invariant under such a deformation. This suggests that our analytic torsion can be defined in the background of a generalized metric and should be invariant under its deformation. Indeed, such a definition is possible using the adjoint of $d_{\bar{k}}$ with respect to the Born-Infeld metric. In this way, deformations of the usual metric and that by a B-field are unified.

4. Functorial properties of analytic torsion

In this section, we state the basic functorial properties of analytic torsion for the twisted de Rham complex. These can be established by a generalization of the proofs of the corresponding results for the usual analytic torsion [47, 19, 41] to the $\mathbb{Z}_2$-graded case. We write $d_{\bar{k}}^{\mathcal{E}}=d_{\bar{k}}^{\mathcal{E},H}$, $\Delta_{\bar{k}}^{\mathcal{E}}=\Delta_{\bar{k}}^{\mathcal{E},H}$ and $\eta_{\bar{k}}^{\mathcal{E}}=\eta_{\bar{k}}^{\mathcal{E},H}$ since the dependence on the flux form H is clear.

Proposition 4.1. *Let X be a closed, oriented Riemannian manifold and $\mathcal{E}_1, \mathcal{E}_2$, flat Hermitian vector bundles on X. Suppose H is a closed odd-degree form on X. Then*

$$\tau(X, \mathcal{E}_1 \oplus \mathcal{E}_2, H) = \tau(X, \mathcal{E}_1, H) \otimes \tau(X, \mathcal{E}_2, H)$$

under the canonical identification

$$\det H^\bullet(X, \mathcal{E}_1 \oplus \mathcal{E}_2, H) \cong \det H^\bullet(X, \mathcal{E}_1, H) \otimes \det H^\bullet(X, \mathcal{E}_2, H)$$

induced by the isomorphism $H^\bullet(X, \mathcal{E}_1 \oplus \mathcal{E}_2, H) \cong H^\bullet(X, \mathcal{E}_1, H) \oplus H^\bullet(X, \mathcal{E}_2, H)$.

Proof. On $\Omega^\bullet(X, \mathcal{E}_1 \oplus \mathcal{E}_2) \cong \Omega^\bullet(X, \mathcal{E}_1) \oplus \Omega^\bullet(X, \mathcal{E}_2)$, the operator $d_{\bar{k}}^{\mathcal{E}_1 \oplus \mathcal{E}_2} = d_{\bar{k}}^{\mathcal{E}_1} \oplus d_{\bar{k}}^{\mathcal{E}_2}$ is block-diagonal. Thus the determinant factorizes: $\mathrm{Det}'((d_{\bar{k}}^{\mathcal{E}_1 \oplus \mathcal{E}_2})^\dagger d_{\bar{k}}^{\mathcal{E}_1 \oplus \mathcal{E}_2}) = \mathrm{Det}'((d_{\bar{k}}^{\mathcal{E}_1})^\dagger d_{\bar{k}}^{\mathcal{E}_1})\, \mathrm{Det}'((d_{\bar{k}}^{\mathcal{E}_2})^\dagger d_{\bar{k}}^{\mathcal{E}_2})$. Under the above identification, we can choose the volume elements such that $\eta_{\bar{k}}^{\mathcal{E}_1 \oplus \mathcal{E}_2} = \eta_{\bar{k}}^{\mathcal{E}_1} \otimes \eta_{\bar{k}}^{\mathcal{E}_2}$. Hence the result. $\square$

Proposition 4.2. *Let X be a closed, oriented manifold of dimension n and $\mathcal{E}$, a flat vector bundle on X. Suppose H is a closed odd-degree form on X. Then*

$$\tau(X, \mathcal{E}, H) = \tau(X, \mathcal{E}^*, H)^{(-1)^{n+1}}$$

under the canonical identification

$$\det H^\bullet(X, \mathcal{E}^*, H) \cong \det H^\bullet(X, \mathcal{E}, H)^{(-1)^{n+1}}$$

induced by the isomorphism $H^\bullet(X, \mathcal{E}^*, H) \cong H^{n-\bullet}(X, \mathcal{E}, H)^*$.

Proof. Using $(d_{\bar{k}}^{\mathcal{E}})^\dagger = \pm \Gamma^{-1} \circ d_{\overline{n+1-k}}^{\mathcal{E}^*} \circ \Gamma$, where $\Gamma \colon \Omega^\bullet(X, \mathcal{E}) \to \Omega^\bullet(X, \mathcal{E}^*)$ is an isometry, the non-zero spectrum of $(d_{\bar{k}}^{\mathcal{E}})^\dagger d_{\bar{k}}^{\mathcal{E}}$, counting multiplicity, is identical to that of $(d_{\overline{n+1-k}}^{\mathcal{E}^*})^\dagger d_{\overline{n+1-k}}^{\mathcal{E}^*}$, and so is the regularized determinant. By the same isometry, the volume elements $\eta_{\bar{k}}^{\mathcal{E}} = \eta_{\overline{n+1-k}}^{\mathcal{E}^*}$ under the above identification. The result follows from the definition of torsion. $\square$

Corollary 4.3. *Let X be a closed, oriented manifold of even dimension and $\mathcal{E}$, a flat acyclic vector bundle on X. Suppose H is a closed odd-degree form on X. Then*

$$\tau(X, \mathcal{E}, H)\, \tau(X, \mathcal{E}^*, H) = 1.$$

In particular, if $\mathcal{E}$ is associated with an orthogonal representation of $\pi_1(X)$ and is acyclic, then

$$\tau(X, \mathcal{E}, H) = 1.$$

Proof. The first result follows directly from Proposition 4.2. If $\mathcal{E}$ is moreover associated with an orthogonal representation, then $\mathcal{E}^* \cong \mathcal{E}$ and $\tau(X, \mathcal{E}, H) = 1$, since the zeta-function regularized determinants are positive numbers. $\square$

Proposition 4.4. *Let X_1, X_2 be two closed oriented manifolds with the same universal covering manifold. Suppose the fundamental group $\pi_1(X_1)$ is a subgroup of $\pi_1(X_2)$. Let ρ_1 be a representation of $\pi_1(X_1)$ and let ρ_2 be the induced representation of $\pi_1(X_2)$. Denote by the flat vector bundles associated with ρ_1, ρ_2 by $\mathcal{E}_1$, $\mathcal{E}_2$, respectively. Suppose the closed odd-degree forms H_1 on X_1 and H_2 on X_2 pull-back to the same form on the universal covering. Then*

$$\tau(X_1, \mathcal{E}_1, H_1) = \tau(X_2, \mathcal{E}_2, H_2)$$

under the canonical identification

$$\det H^\bullet(X_1,\mathcal{E}_1,H_1)\cong\det H^\bullet(X_2,\mathcal{E}_2,H_2)$$

induced by the isomorphism $H^\bullet(X_1,\mathcal{E}_1,H_1)\cong H^\bullet(X_2,\mathcal{E}_2,H_2)$.

Proof. By Theorem 3.4, we can choose the Riemannian metrics on X_1 and X_2 so that they pull-back to the same metric on the universal covering and the Hermitian metrics on $\mathcal{E}_1$ and $\mathcal{E}_2$ associated to metric on the space of representation ρ_1 and the induced metric on the space of representation ρ_2. Then the canonical isomorphism $\Omega^\bullet(X_1,\mathcal{E}_1,H_1)\cong\Omega^\bullet(X_2,\mathcal{E}_2,H_2)$ is an isometry. Following the proof Theorem 2.6 in [47], we deduce that the spectrums, and hence the regularized determinants of $(d^{\mathcal{E}_1}_{\bar k})^\dagger d^{\mathcal{E}_1}_{\bar k}$ and $(d^{\mathcal{E}_2}_{\bar k})^\dagger d^{\mathcal{E}_2}_{\bar k}$ coincide. The volume elements of $H^\bullet(X_1,\mathcal{E}_1,H_1)$ and $H^\bullet(X_2,\mathcal{E}_2,H_2)$ also coincide under the isomorphism. □

Proposition 4.5. *Let X_i $(i=1,2)$ be closed, oriented manifolds and $p_i\colon\mathcal{E}_i\to X_i$ $(i=1,2)$, flat vector bundles. Denote by $\pi_i\colon X_1\times X_2\to X_i$ $(i=1,2)$ the projections. Suppose for $i=0,1$, H_i is a closed odd-degree form on X_i. Set $\mathcal{E}_1\boxtimes\mathcal{E}_2=\pi_1^*\mathcal{E}_1\otimes\pi_2^*\mathcal{E}_2$, $H_1\boxplus H_2=\pi_1^*H_1+\pi_2^*H_2$. Then*

$$\tau(X_1\times X_2,\mathcal{E}_1\boxtimes\mathcal{E}_2,H_1\boxplus H_2)=\tau(X_1,\mathcal{E}_1,H_1)^{\otimes\chi(X_2,\mathcal{E}_2)}\otimes\tau(X_2,\mathcal{E}_2,H_2)^{\otimes\chi(X_1,\mathcal{E}_1)}$$

under the canonical identification

$$\begin{aligned}\det H^\bullet(X_1\times X_2,&\mathcal{E}_1\boxtimes\mathcal{E}_2,H_1\boxplus H_2)\\&\cong(\det H^\bullet(X_1,\mathcal{E}_1,H_1))^{\otimes\chi(X_2,\mathcal{E}_2)}\otimes(\det H^\bullet(X_2,\mathcal{E}_2,H_2))^{\otimes\chi(X_1,\mathcal{E}_1)}\end{aligned}$$

induced by the isomorphism $H^{\bar k}(X_1\times X_2,\mathcal{E}_1\boxtimes\mathcal{E}_2,H_1\boxplus H_2)\cong\bigoplus_{l=0,1}H^{\bar l}(X_1,\mathcal{E}_1,H_1)\otimes H^{\overline{k-l}}(X_2,\mathcal{E}_2,H_2)$, $k=0,1$.

Proof. For $k=0,1$, the space $\Omega^{\bar k}(X_1\times X_2,\mathcal{E}_1\boxtimes\mathcal{E}_2)$ has a dense subspace that is isomorphic to $\bigoplus_{l=0,1}\Omega^{\bar l}(X_1,\mathcal{E}_1)\otimes\Omega^{\overline{k-l}}(X_2,\mathcal{E}_2)$. Under this identification, the operators $d^{\mathcal{E}_1\boxtimes\mathcal{E}_2}=d^{\mathcal{E}_1}\otimes1+1\otimes d^{\mathcal{E}_2}$, $(d^{\mathcal{E}_1\boxtimes\mathcal{E}_2})^\dagger=(d^{\mathcal{E}_1})^\dagger\otimes1+1\otimes(d^{\mathcal{E}_2})^\dagger$, $\Delta^{\mathcal{E}_1\boxtimes\mathcal{E}_2}=\Delta^{\mathcal{E}_1}\otimes1+1\otimes\Delta^{\mathcal{E}_2}$ on $\Omega^\bullet(X_1\times X_2,\mathcal{E}_1\boxtimes\mathcal{E}_2)$. We have

$$\mathrm{spec}'(\Delta^{\mathcal{E}_1\boxtimes\mathcal{E}_2}_{\bar0})=\{\lambda_1+\lambda_2>0\,|\,\lambda_i\in\mathrm{spec}(\Delta^{\mathcal{E}_i}_{\bar0})\text{ or }\lambda_i\in\mathrm{spec}(\Delta^{\mathcal{E}_i}_{\bar1}),\ i=1,2\}$$

and therefore by (4),

(8)

$$\begin{aligned}&\sum_{k=0,1}(-1)^k\zeta\big(s,(d^{\mathcal{E}_1\boxtimes\mathcal{E}_2}_{\bar k})^\dagger d^{\mathcal{E}_1\boxtimes\mathcal{E}_2}_{\bar k}\big)\\&=\sum_{k=0,1}\left(\sum_{\substack{\lambda_i\in\mathrm{spec}(\Delta^{\mathcal{E}_i}_{\bar k}),i=1,2\\\lambda_1+\lambda_2\in\mathrm{spec}_{\mathrm{I}}(\Delta^{\mathcal{E}_1\boxtimes\mathcal{E}_2}_{\bar0})}}\frac{m_{\mathrm{I}}(\lambda_1+\lambda_2,\Delta^{\mathcal{E}_1\boxtimes\mathcal{E}_2}_{\bar0})}{(\lambda_1+\lambda_2)^s}-\sum_{\substack{\lambda_i\in\mathrm{spec}(\Delta^{\mathcal{E}_i}_{\bar k}),i=1,2\\\lambda_1+\lambda_2\in\mathrm{spec}_{\mathrm{II}}(\Delta^{\mathcal{E}_1\boxtimes\mathcal{E}_2}_{\bar0})}}\frac{m_{\mathrm{II}}(\lambda_1+\lambda_2,\Delta^{\mathcal{E}_1\boxtimes\mathcal{E}_2}_{\bar0})}{(\lambda_1+\lambda_2)^s}\right).\end{aligned}$$

We now show that total contribution to the sum (8) from (λ_1,λ_2) vanishes if both $\lambda_1,\lambda_2>0$. We consider four cases.

1. If both $\lambda_i\in\mathrm{spec}_{\mathrm{I}}(\Delta^{\mathcal{E}_i}_{\bar0})$, then there is a non-zero $\omega_i\in\mathrm{im}(d^{\mathcal{E}_i}_{\bar0})^\dagger$ such that $\Delta^{\mathcal{E}_i}_{\bar0}\omega_i=\lambda_i\omega_i$ for each $i=1,2$. It is easy to see that $\omega_1\otimes\omega_2\in\mathrm{im}(d^{\mathcal{E}_1\boxtimes\mathcal{E}_2}_{\bar0})^\dagger$ is an

eigenvector of $\Delta_{\bar 0}^{\mathcal{E}_1\boxtimes\mathcal{E}_2}$ with eigenvalue $\lambda_1+\lambda_2\in\mathrm{spec}_{\mathrm{I}}(\Delta_{\bar 0}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})$. On the other hand, $d_{\bar 0}^{\mathcal{E}_i}\omega_i$ is a (non-zero) eigenvector of $\Delta_{\bar i}^{\mathcal{E}_i}$ with eigenvalue λ_i. Hence $\lambda_i\in\mathrm{spec}'(\Delta_{\bar 1}^{\mathcal{E}_i})$. Since $d_{\bar 0}^{\mathcal{E}_1}\omega_1\otimes d_{\bar 0}^{\mathcal{E}_2}\omega_2\in\mathrm{im}(d_{\bar 1}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})$, we also have $\lambda_1+\lambda_2\in\mathrm{spec}_{\mathrm{II}}(\Delta_{\bar 0}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})$. So the contribution of $(\lambda_1,\lambda_2)\in\mathrm{spec}'(\Delta_{\bar 0}^{\mathcal{E}_1})\times\mathrm{spec}'(\Delta_{\bar 0}^{\mathcal{E}_2})$ cancels that of $(\lambda_1,\lambda_2)\in\mathrm{spec}'(\Delta_{\bar 1}^{\mathcal{E}_1})\times\mathrm{spec}'(\Delta_{\bar 1}^{\mathcal{E}_2})$.
2. Similarly, if both $\lambda_i\in\mathrm{spec}_{\mathrm{II}}(\Delta_{\bar 0}^{\mathcal{E}_i})$ $(i=1,2)$, the corresponding contribution is also canceled.
3. If $\lambda_1\in\mathrm{spec}_{\mathrm{I}}(\Delta_{\bar 0}^{\mathcal{E}_1})$ but $\lambda_2\in\mathrm{spec}_{\mathrm{II}}(\Delta_{\bar 0}^{\mathcal{E}_2})$, let ω_i $(i=1,2)$ be the corresponding eigenvectors of $\Delta_{\bar 0}^{\mathcal{E}_i}$. Then $\omega_1\otimes\omega_2$ and $d_{\bar 0}^{\mathcal{E}_1}\omega_1\otimes(d_{\bar 1}^{\mathcal{E}_2})^\dagger\omega_2$ are linearly independent eigenvectors of $\Delta_{\bar 0}^{\mathcal{E}_1\boxtimes\mathcal{E}_2}$ with the same eigenvalue $\lambda_1+\lambda_2$. It is easy to see that one linear combination $\omega_1\otimes\omega_2-\lambda_1^{-1}d_{\bar 0}^{\mathcal{E}_1}\omega_1\otimes(d_{\bar 1}^{\mathcal{E}_2})^\dagger\omega_2$ is in $\mathrm{im}(d_{\bar 0}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})^\dagger$, yielding $\lambda_1+\lambda_2\in\mathrm{spec}_{\mathrm{I}}(\Delta_{\bar 0}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})$ while another (independent) combination $\omega_1\otimes\omega_2+\lambda_2^{-1}d_{\bar 0}^{\mathcal{E}_1}\omega_1\otimes(d_{\bar 1}^{\mathcal{E}_2})^\dagger\omega_2$ is in $\mathrm{im}(d_{\bar 1}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})$, yielding $\lambda_1+\lambda_2\in\mathrm{spec}_{\mathrm{II}}(\Delta_{\bar 0}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})$. So the contributions of (λ_1,λ_2) also cancel in this case.
4. The case $\lambda_1\in\mathrm{spec}_{\mathrm{II}}(\Delta_{\bar 0}^{\mathcal{E}_1})$, $\lambda_2\in\mathrm{spec}_{\mathrm{I}}(\Delta_{\bar 0}^{\mathcal{E}_2})$ is similar.

The non-zero contributions to (8) are thus from the subspaces $\mathrm{im}(d_{\bar k}^{\mathcal{E}_1})^\dagger\otimes\ker(\Delta_{\bar l}^{\mathcal{E}_2})$, $\ker(\Delta_{\bar l}^{\mathcal{E}_1})\otimes\mathrm{im}(d_{\bar k}^{\mathcal{E}_2})^\dagger\subset\mathrm{im}(d_{\overline{k+l}}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})^\dagger$, $k,l=0,1$. Since $\dim\ker(\Delta_{\bar l}^{\mathcal{E}_i})=b_{\bar l}(X_i,\mathcal{E}_i,H_i)$, we have

$$\begin{aligned}
&\sum_{k=0,1}(-1)^k\zeta\big(s,(d_{\bar k}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})^\dagger d_{\bar k}^{\mathcal{E}_1\boxtimes\mathcal{E}_2}\big)\\
&=\sum_{k,l=0,1}(-1)^{k+l}\left(\sum_{\lambda_1\in\mathrm{spec}'((d_{\bar k}^{\mathcal{E}_1})^\dagger d_{\bar k}^{\mathcal{E}_1})}\frac{m\big(\lambda_1,(d_{\bar k}^{\mathcal{E}_1})^\dagger d_{\bar k}^{\mathcal{E}_1}\big)\,b_{\bar l}(X_2,\mathcal{E}_2,H_2)}{\lambda_1^s}\right.\\
&\qquad\qquad\left.+\sum_{\lambda_2\in\mathrm{spec}'((d_{\bar k}^{\mathcal{E}_2})^\dagger d_{\bar k}^{\mathcal{E}_2})}\frac{m\big(\lambda_2,(d_{\bar k}^{\mathcal{E}_2})^\dagger d_{\bar k}^{\mathcal{E}_2}\big)\,b_{\bar l}(X_1,\mathcal{E}_1,H_1)}{\lambda_2^s}\right)\\
&=\chi(X_2,\mathcal{E}_2)\sum_{k=0,1}(-1)^k\zeta\big(s,(d_{\bar k}^{\mathcal{E}_1})^\dagger d_{\bar k}^{\mathcal{E}_1}\big)+\chi(X_1,\mathcal{E}_1)\sum_{k=0,1}(-1)^k\zeta\big(s,(d_{\bar k}^{\mathcal{E}_2})^\dagger d_{\bar k}^{\mathcal{E}_2}\big)
\end{aligned}$$

and therefore

$$\frac{\mathrm{Det}'(d_{\bar 0}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})^\dagger d_{\bar 0}^{\mathcal{E}_1\boxtimes\mathcal{E}_2}}{\mathrm{Det}'(d_{\bar 1}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})^\dagger d_{\bar 1}^{\mathcal{E}_1\boxtimes\mathcal{E}_2}}=\left(\frac{\mathrm{Det}'(d_{\bar 0}^{\mathcal{E}_1})^\dagger d_{\bar 0}^{\mathcal{E}_1}}{\mathrm{Det}'(d_{\bar 1}^{\mathcal{E}_1})^\dagger d_{\bar 1}^{\mathcal{E}_1}}\right)^{\chi(X_2,\mathcal{E}_2)}\left(\frac{\mathrm{Det}'(d_{\bar 0}^{\mathcal{E}_2})^\dagger d_{\bar 0}^{\mathcal{E}_2}}{\mathrm{Det}'(d_{\bar 1}^{\mathcal{E}_2})^\dagger d_{\bar 1}^{\mathcal{E}_2}}\right)^{\chi(X_1,\mathcal{E}_1)}.$$

For the volume elements, we can choose

$$\eta_{\bar k}^{\mathcal{E}_1\boxtimes\mathcal{E}_2}=\bigotimes_{l=0,1}(\eta_{\bar l}^{\mathcal{E}_1})^{\otimes b_{\overline{k-l}}(X_2,\mathcal{E}_2,H_2)}\otimes(\eta_{\bar l}^{\mathcal{E}_2})^{\otimes b_{\overline{k-l}}(X_1,\mathcal{E}_1,H_1)}$$

and hence

$$\eta_{\bar 0}^{\mathcal{E}_1\boxtimes\mathcal{E}_2}\otimes(\eta_{\bar 1}^{\mathcal{E}_1\boxtimes\mathcal{E}_2})^{-1}=(\eta_{\bar 0}^{\mathcal{E}_1}\otimes(\eta_{\bar 1}^{\mathcal{E}_1})^{-1})^{\otimes\chi(X_2,\mathcal{E}_2)}\otimes(\eta_{\bar 0}^{\mathcal{E}_2}\otimes(\eta_{\bar 1}^{\mathcal{E}_2})^{-1})^{\otimes\chi(X_1,\mathcal{E}_1)}.$$

□

It would be interesting to establish the behavior of the analytic torsion (form) under a general smooth fibration, analogous to [7, 21, 36], for the twisted de Rham or other $\mathbb{Z}_2$-graded complexes.

5. Calculations of analytic torsion and the simplicial analogue

5.1. **Analytic torsion when the flux is a top-degree form.** Recall that the twisted cohomology groups can be computed by the spectral sequence in §1.2. In this process, each complex $(E_r^\bullet,\delta_r)$ is finite dimensional for $r\ge 2$ when X is compact. The Knudsen-Mumford isomorphisms [32] $\det E_r^\bullet\cong\det E_{r+1}^\bullet$ for $r\ge 2$ yield an isomorphism

$$\kappa\colon\ \det H^\bullet(X,\mathcal{E})\to\det H^\bullet(X,\mathcal{E},H) \tag{9}$$

since the spectral sequence converges to the twisted cohomology.

Proposition 5.1. *Suppose X is a compact oriented manifold of odd dimension and $\mathcal{E}$ is a flat vector bundle associated to an orthogonal or unitary representation of $\pi_1(X)$. Assume $n=\dim X>1$ and let H be an n-form on X. Then*

$$\tau(X,\mathcal{E},H)=\kappa(\tau(X,\mathcal{E})).$$

Proof. By Theorem 3.4, we can choose a Riemannian metric on X so that $\mathrm{vol}(X)=1$; let $\nu=*1$ be the volume form on X. By Theorem 3.8, we can also assume that $H=[H]\,\nu$, where $[H]\in H^n(X,\mathbb{R})\cong\mathbb{R}$ is a real number. If $[H]=0$, then the statement is trivial; we assume that $[H]\ne 0$. Since $\mathcal{E}$ is a flat vector bundle associated to an orthogonal or unitary representation, we have $H^0(X,\mathcal{E})\cong\overline{H^n(X,\mathcal{E})}^*$; let $b_0:=\dim H^0(X,\mathcal{E})=\dim H^n(X,\mathcal{E})$. Let η_i be the unit volume element of $H^i(X,\mathcal{E})$ for $0\le i\le n$. The metric-independent isomorphism (9) is given by

$$\kappa\colon\ \bigotimes_{i=0}^n\eta_i^{(-1)^i}\mapsto|[H]|^{b_0}\,\eta_{\bar 0}\otimes\eta_{\bar 1}^{-1}.$$

Let d_i $(0\le i\le n-1)$ be the differential on $C^i=\Omega^i(X,\mathcal{E})$. Then $d_{\bar k}$ is equal to $\left(\begin{smallmatrix}d_0&0\\H&d_{n-1}\end{smallmatrix}\right)$ on $C^0\oplus C^{n-1}$, d_i on C^i for $i\le i\le n-2$, and 0 on C^n. Here H also stands for taking wedge product with H. The Ray-Singer torsion is

$$\tau(X,\mathcal{E})=\prod_{i=0}^{n-1}(\mathrm{Det}'\,d_i^\dagger d_i)^{(-1)^i/2}\bigotimes_{i=0}^n\eta_i^{(-1)^i}$$

while the torsion for the twisted de Rham complex is

$$\tau(X,\mathcal{E},H)=\mathrm{Det}'\begin{pmatrix}d_0^\dagger d_0+H^\dagger H & H^\dagger d_{n-1}\\ d_{n-1}^\dagger H & d_{n-1}^\dagger d_{n-1}\end{pmatrix}^{1/2}\prod_{i=1}^{n-2}(\mathrm{Det}'\,d_i^\dagger d_i)^{(-1)^i/2}\,\eta_{\bar 0}\otimes\eta_{\bar 1}^{-1}.$$

The result follows from the following Lemma. □

Lemma 5.2. *Under the above assumptions, we have*

$$\mathrm{Det}'\begin{pmatrix}d_0^\dagger d_0+H^\dagger H & H^\dagger d_{n-1}\\ d_{n-1}^\dagger H & d_{n-1}^\dagger d_{n-1}\end{pmatrix}=[H]^{2b_0}\,\mathrm{Det}'\,d_0^\dagger d_0\,\mathrm{Det}'\,d_{n-1}^\dagger d_{n-1}.$$

Proof. Let Q_i be the orthogonal projection from (the completion of) C^i onto $\ker(\Delta_i)$, $0\le i\le n$. Set $\tilde\Delta_{n-1}=\Delta_{n-1}+Q_{n-1}$. Then

$$\mathrm{Det}'\begin{pmatrix}d_0^\dagger d_0+H^\dagger H & H^\dagger d_{n-1}\\ d_{n-1}^\dagger H & d_{n-1}^\dagger d_{n-1}\end{pmatrix}=\mathrm{Det}'\begin{pmatrix}\Delta_0+[H]^2 & H^\dagger d_{n-1}\\ d_{n-1}^\dagger H & \tilde\Delta_{n-1}\end{pmatrix}(\mathrm{Det}'\,d_{n-2}^\dagger d_{n-2})^{-1}.$$

Since $d_{n-1}\tilde{\Delta}_{n-1}^{-1}d_{n-1}^\dagger = 1 - Q_n$ and hence $H^\dagger d_{n-1}\tilde{\Delta}_{n-1}^{-1}d_{n-1}^\dagger H = [H]^2(1-Q_0)$, we have

$$\begin{aligned}\begin{pmatrix}\Delta_0+[H]^2 & H^\dagger d_{n-1}\\ d_{n-1}^\dagger H & \tilde{\Delta}_{n-1}\end{pmatrix} &= \begin{pmatrix}\Delta_0+[H]^2Q_0 & H^\dagger d_{n-1}\\ 0 & \tilde{\Delta}_{n-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ \tilde{\Delta}_{n-1}^{-1}d_{n-1}^\dagger H & 1\end{pmatrix}\\ &= \begin{pmatrix}1 & H^\dagger d_{n-1}\tilde{\Delta}_{n-1}^{-1}\\ 0 & 1\end{pmatrix}\begin{pmatrix}\Delta_0+[H]^2Q_0 & 0\\ 0 & \tilde{\Delta}_{n-1}\end{pmatrix}\begin{pmatrix}1 & 0\\ \tilde{\Delta}_{n-1}^{-1}d_{n-1}^\dagger H & 1\end{pmatrix}.\end{aligned}$$

We note that $n = \dim X$ is odd. Since the elliptic pseudo-differential operators

$$\begin{pmatrix}\Delta_0+[H]^2 & H^\dagger d_{n-1}\\ d_{n-1}^\dagger H & \tilde{\Delta}_{n-1}\end{pmatrix},\quad \begin{pmatrix}\Delta_0+[H]^2Q_0 & H^\dagger d_{n-1}\\ 0 & \tilde{\Delta}_{n-1}\end{pmatrix},\quad \begin{pmatrix}\Delta_0+[H]^2Q_0 & 0\\ 0 & \tilde{\Delta}_{n-1}\end{pmatrix}$$

of order 2 on $C^0 \oplus C^{n-1}$ are odd in the sense of Kontsevich and Vishik [33] and are invertible, the pseudo-differential operators

$$\begin{pmatrix}1 & H^\dagger d_{n-1}\tilde{\Delta}_{n-1}^{-1}\\ 0 & 1\end{pmatrix},\quad \begin{pmatrix}1 & 0\\ \tilde{\Delta}_{n-1}^{-1}d_{n-1}^\dagger H & 1\end{pmatrix}$$

of order 0 are also odd and their determinants are defined [33]; we will denote these determinants by det to distinguish them from zeta-function regularized determinants Det'. In fact, for any $a > 0$,

$$\mathrm{Det}'\begin{pmatrix}\Delta_0+a & H^\dagger d_{n-1}\\ 0 & \tilde{\Delta}_{n-1}\end{pmatrix} = \det\begin{pmatrix}1 & H^\dagger d_{n-1}\tilde{\Delta}_{n-1}^{-1}\\ 0 & 1\end{pmatrix}\ \mathrm{Det}'\begin{pmatrix}\Delta_0+a & 0\\ 0 & \tilde{\Delta}_{n-1}\end{pmatrix}.$$

Choosing a such that the spectrums of $\Delta_0 + a$ and $\tilde{\Delta}_{n-1}$ are disjoint, the operators

$$\begin{pmatrix}\Delta_0+a & H^\dagger d_{n-1}\\ 0 & \tilde{\Delta}_{n-1}\end{pmatrix},\quad \begin{pmatrix}\Delta_0+a & 0\\ 0 & \tilde{\Delta}_{n-1}\end{pmatrix}$$

have identical spectrums and hence the same zeta-function regularized determinant. Thus

$$\det\begin{pmatrix}1 & H^\dagger d_{n-1}\tilde{\Delta}_{n-1}^{-1}\\ 0 & 1\end{pmatrix} = 1$$

and, similarly,

$$\det\begin{pmatrix}1 & 0\\ \tilde{\Delta}_{n-1}^{-1}d_{n-1}^\dagger H & 1\end{pmatrix} = 1.$$

As determinants factorize for odd pseudo-differential operators of non-negative order on an odd-dimensional manifold [33], we get

$$\begin{aligned}\mathrm{Det}'\begin{pmatrix}\Delta_0+[H]^2 & H^\dagger d_{n-1}\\ d_{n-1}^\dagger H & \tilde{\Delta}_{n-1}\end{pmatrix} &= \mathrm{Det}'\begin{pmatrix}\Delta_0+[H]^2Q_0 & 0\\ 0 & \tilde{\Delta}_{n-1}\end{pmatrix}\\ &= [H]^{2b_0}\ \mathrm{Det}'\,d_0^\dagger d_0\ \mathrm{Det}'\,d_{n-1}^\dagger d_{n-1}\ \mathrm{Det}'\,d_{n-2}^\dagger d_{n-2}\end{aligned}$$

and the result follows. □

We note that neither Lemma 5.2 nor Proposition 5.1 is valid if $\dim X = 1$ and $[H] \neq 0$. We give a heuristic explanation of Lemma 5.2 when $n > 1$. For any $\lambda \in \mathrm{spec}'(d_0^\dagger d_0)$, let ω_λ be an eigenvector corresponding to λ. Then $*d_0\omega_\lambda/\sqrt{\lambda}$ is an eigenvector of $d_{n-1}^\dagger d_{n-1}$ with the same eigenvalue. On the subspace spanned by

ω_λ and $*d_0\omega_\lambda/\sqrt{\lambda}$, the operator $d_{\bar 0}^\dagger d_{\bar 0}$ acts as $\left(\begin{smallmatrix}\frac{\lambda+[H]^2}{[H]\sqrt{\lambda}} & [H]\sqrt{\lambda}\\ & \lambda\end{smallmatrix}\right)$, whose determinant is λ^2. Notice that

$$C^0\oplus\operatorname{im}(d_{n-1}^\dagger)=\ker(\Delta_0)\oplus\bigoplus_{\lambda\in\operatorname{spec}'(d_0^\dagger d_0)}\operatorname{span}_{\mathbb{C}}\{\omega_\lambda,*d_0\omega_\lambda/\sqrt{\lambda}\}$$

and $\ker(\Delta_0)$ is in the eigenspace of $\Delta_0+[H]^2Q_0$ corresponding to the eigenvalue $[H]^2$ (with multiplicity b_0). The "product" of these λ^2 together with $[H]^2$ leads to the result.

Under the assumptions of Proposition 5.1, $\zeta(0,d_{\bar 1}^\dagger d_{\bar 1})$ for any H is the same as its value when $H=0$; it would be interesting to find the value of $\zeta(0,d_{\bar 0}^\dagger d_{\bar 0})$ when $[H]\neq 0$. (See Corollary 3.6 and the discussion that follows.)

In addition to κ in (9), there is another natural isomorphism κ_0 which maps between the alternating products of unit volume elements, i.e.,

$$\kappa_0\colon\bigotimes_{i=0}^n\eta_i^{(-1)^i}\mapsto\eta_{\bar 0}\otimes\eta_{\bar 1}^{-1}.$$

If H is a top-degree form as in Proposition 5.1, then κ_0 is independent of the choice of metrics on X and on $\mathcal{E}$. The appearance of $|[H]|$ in

$$\tau(X,\mathcal{E},H)=|[H]|^{b_0}\,\kappa_0(\tau(X,\mathcal{E}))$$

is consistent with the metric invariance of both $\tau(X,\mathcal{E})$ and $\tau(X,\mathcal{E},H)$ and dependence of the latter on the cohomology class $[H]$ only.

Proposition 5.1 applies especially to 3-dimensional manifolds because H is automatically a top-degree form if it contains no 1-form (which can be absorbed in the flat connection). The Ray-Singer torsion has been calculated explicitly, directly or with the help of the Cheeger-Müller theorem, for many 3-manifolds including lens spaces [46, 22] and compact hyperbolic manifolds [23]. As a consequence, we get many non-trivial examples of analytic torsion for the twisted de Rham complexes of 3-manifolds.

5.2. **Simplicial analogue of the torsion in a special case.** One of the standard ways to compute the classical Ray-Singer torsion is to use the Cheeger-Müller theorem, relating it to the Reidemeister torsion. Although there is difficulty in defining the simplicial counterpart of the twisted analytic torsion in general, we will be able to do so under the condition that the degree of the flux form is sufficiently high.

We first recall the construction of the Reidemeister torsion (cf. [41]). Suppose the manifold X is equipped with a smooth triangularization or a CW complex structure. Let $(C_\bullet(K),\partial)$ be the chain complex of the simplicial or cellular complex K with real coefficients. Choose an embedding of K as a fundamental domain in the corresponding complex $\tilde K$ of the universal covering space $\tilde X$. Then each $C_i(\tilde K)$ ($0\le i\le n$, where $n=\dim X$) is a free module over the group algebra $\mathbb{R}[\pi_1(X)]$ and the i-cells of K form a basis. Given a finite dimensional representation $\rho\colon\pi_1(X)\to\mathrm{GL}(E)$, we define a cochain complex

$$C^\bullet(K,E):=\operatorname{Hom}_{\mathbb{R}[\pi_1(X)]}(C_\bullet(\tilde K),E)$$

with coboundary map ∂^*, whose cohomology is denoted by $H^\bullet(K,E)$. With an Hermitian form on E, we choose a unit volume element of E. This, together with the basis dual to the i-cells in K, defines a volume element $\mu_i\in\det C^i(K,E)$. We assume that the representation ρ is unimodular. Unimodularity means that

$|\det \rho(\gamma)| = 1$ for all $\gamma \in \pi_1(X)$. Then the volume element μ_i is, up to a phase, independent of the choice of the embedding of K in $\tilde{K}$. The *Reidemeister torsion* or *R-torsion* $\tau(K,E) \in \det H^\bullet(K,E)$ is defined as the image of $\otimes_{i=0}^n \mu_i^{(-1)^i}$ under the isomorphism $\det C^\bullet(K,E) \cong \det H^\bullet(K,E)$. It is invariant under subdivisions of the complex K. If X is odd-dimensional, the Euler number $\chi(K) = 0$, and $\tau(K,E)$ (up to a phase) does not depend on the choice of the Hermitian form on E. By the de Rham theorem, $H^\bullet(X,\mathcal{E}) \cong H^\bullet(K,E)$ and hence $\det H^\bullet(X,\mathcal{E}) \cong \det H^\bullet(K,E)$. The theorem of Cheeger and Müller [19, 40, 41] states that $\tau(X,\mathcal{E}) = \tau(K,E)$ under the above identification.

Recall that the cup product at the cochain level is associative but not graded commutative. We now assume that each homogeneous component of H is of degree greater than $\dim X/2 = n/2$. Let $h \in C^{\bar{1}}(K,E)$ be a representative of $[H] \in H^{\bar{1}}(X,\mathcal{E}) \cong H^{\bar{1}}(K,E)$. Then since $h \cup h = 0$, we have a $\mathbb{Z}_2$-graded cochain complex $(C^\bullet(K,E), \partial_h^*)$, where $\partial_h^* = \partial^* + h \cup \cdot$. Denote its cohomology groups by $H^{\bar{k}}(K,E,h)$, $k = 0,1$. There is then an isomorphism $\det C^\bullet(K,E) \cong \det H^\bullet(K,E,h)$. We define the twisted version of the R-torsion $\tau(K,E,h)$ as the image of $\otimes_{i=0}^n \mu_i^{(-1)^i}$ under the above isomorphism. This will be the simplicial counterpart of the analytic torsion $\tau(X,\mathcal{E},H)$.

Lemma 5.3. *There is a canonical isomorphism $H^\bullet(X,\mathcal{E},H) \cong H^\bullet(K,E,h)$.*

Proof. Just as $\Omega^\bullet(X,\mathcal{E})$, the complex $C^\bullet(K,E)$ has a filtration

$$F^p C^{\bar{k}}(K,E) = \bigoplus_{\substack{i \geq p \\ i = k \bmod 2}} C^i(K,E),$$

which yields a spectral sequence $\{'E_r^{pq}, \delta'_r\}$ converging to $H^\bullet(K,E,h)$. The cochain map $\Omega^\bullet(X,\mathcal{E}) \to C^\bullet(K,E)$ that induces the de Rham isomorphism preserves the filtrations. Therefore there is a morphism of the the spectral sequences $\{E_r^{pq}, \delta_r\} \to \{'E_r^{pq}, \delta'_r\}$. By the de Rham theorem, this morphism is an isomorphism starting with the E_2-terms, which implies the result. □

We have the following analogue of the Cheeger-Müller theorem when H or h is of top degree.

Theorem 5.4. *With the same assumptions of Proposition 5.1 and under identification given by Lemma 5.3, we have*

$$\tau(X,\mathcal{E},H) = \tau(K,E,h).$$

Proof. Let

$$\kappa' \colon \det H^\bullet(K,E) \to \det H^\bullet(K,E,h)$$

be the isomorphism induced by the Knudsen-Mumford isomorphisms in the spectral sequence $\{'E_r^{pq}\}$. The morphism of the two spectral sequences in the proof of Lemma 5.3 induces a commutative diagram

$$\begin{array}{ccc} \det H^\bullet(X,\mathcal{E}) & \xrightarrow{\kappa} & \det H^\bullet(X,\mathcal{E},H) \\ \cong\downarrow & & \cong\downarrow \\ \det H^\bullet(K,E) & \xrightarrow{\kappa'} & \det H^\bullet(K,E,h). \end{array}$$

By Proposition 5.1, we have $\tau(X,\mathcal{E},H) = \kappa(\tau(X,\mathcal{E}))$. On the other hand, it is clear from the definition of $\tau(K,E,h)$ that $\tau(K,E,h) = \kappa'(\tau(K,E))$. The results follows

from the Cheeger-Müller theorem $\tau(X,\mathcal{E}) = \tau(K,E)$ since the representation is orthogonal or unitary. $\square$

Consider for example the lens space $X = L(1,p)$, $p \in \mathbb{Z}$. It has a cellular structure K with one i-cell e_i for each $i = 0,1,2,3$. On the dual basis e_i^* $(0 \le i \le 3)$, we have

$$\partial^* e_0^* = 0, \quad \partial^* e_1^* = p\, e_2^*, \quad \partial^* e_2^* = 0, \quad \partial^* e_3^* = 0.$$

So the Reidemeister torsion is $\tau(K) = |p|^{-1}\eta_0 \otimes \eta_3^{-1}$. If $h = q\, e_3^*$, then

$$\partial_h^* e_0^* = q\, e_3^*, \quad \partial_h^* e_1^* = p\, e_2^*, \quad \partial_h^* e_2^* = 0, \quad \partial_h^* e_3^* = 0,$$

and the twisted torsion is $\tau(K,h) = |qp^{-1}|$.

5.3. ***T*-duality for circle bundles and analytic torsion.** Let $\mathbb{T}$ be the circle group. Suppose X is a compact, oriented manifold and is the total space of a principal $\mathbb{T}$-bundle

$$\begin{array}{ccc} \mathbb{T} & \longrightarrow & X \\ & & \big\downarrow{\scriptstyle \pi} \\ & & M \end{array}$$

over a compact, oriented manifold M and H, a closed 3-form on X that has integral periods. The flat vector bundle $\mathcal{E}$ is taken to be the trivial real line bundle with the trivial connection. Let $\hat{\mathbb{T}}$ be the dual circle group. Then the T-dual principal circle bundle [12]

$$\begin{array}{ccc} \hat{\mathbb{T}} & \longrightarrow & \hat{X} \\ & & \big\downarrow{\scriptstyle \hat{\pi}} \\ & & M \end{array}$$

is determined topologically by its first Chern class $c_1(\hat{X}) = \pi_*[H]$. We have the commutative diagram

$$\begin{tikzcd} & X \times_M \hat{X} \arrow[dl, "p"'] \arrow[dr, "\hat{p}"] & \\ X \arrow[dr, "\pi"'] & & \hat{X} \arrow[dl, "\hat{\pi}"] \\ & M & \end{tikzcd}$$

where $X \times_M \hat{X}$ denotes the correspondence space. The Gysin sequence for $\hat{X}$ enables us to define a T-dual flux $[\hat{H}] \in H^3(\hat{X},\mathbb{Z})$ satisfying $c_1(X) = \hat{\pi}_*[\hat{H}]$ and $p^*[H] = \hat{p}^*[\hat{H}] \in H^3(X \times_M \hat{X},\mathbb{Z})$. Thus T-duality for circle bundles exchanges the H-flux on the one side and the Chern class on the other. It can be shown [12] that $H^\bullet(X,H) \cong H^{\bullet+1}(\hat{X},\hat{H})$ and consequently,

$$\det H^\bullet(X,H) \cong (\det H^\bullet(\hat{X},\hat{H}))^{-1}. \tag{10}$$

We wish to explore the relation between the twisted torsions $\tau(X,H) \in \det H^\bullet(X,H)$ and $\tau(\hat{X},\hat{H}) \in \det H^\bullet(\hat{X},\hat{H})$ under the above identification.

We next explain T-duality at the level of differential forms. Choosing connection 1-forms A and $\hat{A}$ on the circle bundles X and $\hat{X}$, we define the metrics on X and $\hat{X}$ by

$$g_X = \pi^* g_M + A \odot A, \quad g_{\hat{X}} = \hat{\pi}^* g_M + \hat{A} \odot \hat{A},$$

respectively. We assume, without loss of generality, that H is a $\mathbb{T}$-invariant 3-form on X. Denote by $F, \hat{F} \in \Omega^2(M)$ the curvature 2-forms of $A, \hat{A}$, respectively. Since $H - A \wedge \pi^* \hat{F}$ is a basic differential form on X, we have $H = A \wedge \pi^* \hat{F} - \pi^* \Omega$ for some $\Omega \in \Omega^3(M)$. Define the T-dual flux $\hat{H}$ by $\hat{H} = \hat{\pi}^* F \wedge \hat{A} - \hat{\pi}^* \Omega$. Then $\hat{H}$ is closed and $\hat{\mathbb{T}}$-invariant. We define linear maps $T\colon \Omega^{\bar{k}}(X) \to \Omega^{\overline{k+1}}(\hat{X})$ for $k = 0, 1$ by

$$T(\omega) = \int_{\mathbb{T}} e^{p^* A \wedge \hat{p}^* \hat{A}} \, p^* \omega, \quad \omega \in \Omega^\bullet(X).$$

Lemma 5.5. *Under the above choices of Riemannian metrics and flux forms,*

$$T\colon \Omega^{\bar{k}}(X)^{\mathbb{T}} \to \Omega^{\overline{k+1}}(\hat{X})^{\hat{\mathbb{T}}},$$

for $k = 0, 1$, are isometries, inducing isometries on the spaces of twisted harmonic forms and hence on the twisted cohomology groups.

Proof. For any $\omega = \pi^* \omega_1 + A \wedge \pi^* \omega_2 \in \Omega^\bullet(X)^{\mathbb{T}}$, where $\omega_1, \omega_2 \in \Omega^\bullet(M)$, we have $T(\omega) = \hat{\pi}^* \omega_2 + \hat{A} \wedge \hat{\pi}^* \omega_1$. The isometry of T follows from

$$\int_X \omega \wedge *_X \, \omega = \int_M \omega_1 \wedge *_M \, \omega_1 + \int_M \omega_2 \wedge *_M \, \omega_2.$$

Since $d(p^* A \wedge \hat{p}^* \hat{A}) = -p^* H + \hat{p}^* \hat{H}$, we have $T \circ d^H = d^{\hat{H}} \circ T$. So T acts on the spaces of twisted harmonic forms and on the twisted cohomology groups. □

When X is a 3-manifold, Proposition 5.1 relates $\tau(X, H)$ to $\tau(X)$, which can be calculated by the spectral sequence of fibration [21, 22, 36].

Proposition 5.6. *Let X be a oriented 3-manifold which a $\mathbb{T}$-fibration over a compact, oriented surface M and H, a flux 3-form on X. Suppose there is a T-dual fibration $\hat{X}$ with flux form $\hat{H}$. Then $\tau(X, H) = \tau(\hat{X}, \hat{H})^{-1}$ under the identification (10).*

Proof. We can choose the metrics and the flux forms on X, $\hat{X}$ as above. Let $p = c_1(X) \in H^2(M, \mathbb{Z}) \cong \mathbb{Z}$ and $q = [H] \in H^3(X, \mathbb{Z}) \cong \mathbb{Z}$. If $p = 0$, then $X = M \times \mathbb{T}$. If $q = 0$ as well, then $\tau(X) = \eta_{\bar{0}}^X \otimes (\eta_{\bar{1}}^X)^{-1}$. If $q \neq 0$, then by Proposition 5.1,

$$\tau(X, H) = |[H]| \, \kappa_0(\tau(X)) = |q| \, \kappa_0(\eta_{\bar{0}}^X \otimes (\eta_{\bar{1}}^X)^{-1}) = |q| \, \eta_{\bar{0}}^{X,H} \otimes (\eta_{\bar{1}}^{X,H})^{-1}.$$

If $p \neq 0$ but $q = 0$, then we can compute $\tau(X)$ by the Gysin sequence of the fibration $X \to M$[36] and get $\tau(X) = |p|^{-1} \eta_{\bar{0}}^X \otimes (\eta_{\bar{1}}^X)^{-1}$. If both $p, q \neq 0$, then again by Proposition 5.1,

(11)
$$\tau(X, H) = |[H]| \, \kappa_0(\tau(X)) = |q| \, \kappa_0(|p|^{-1} \eta_{\bar{0}}^X \otimes (\eta_{\bar{1}}^X)^{-1}) = |qp^{-1}| \, \eta_{\bar{0}}^{X,H} \otimes (\eta_{\bar{1}}^{X,H})^{-1}.$$

The result follows since T-duality interchanges p and q and since the isometries in Lemma 5.5 identify $\eta_{\bar{k}}^{X,H}$ with $\eta_{\overline{k+1}}^{\hat{X},\hat{H}}$ for $k = 0, 1$.

We note that (11) is consistent with the simplicial calculation in §5.2 when $X = L(1, p)$, verifying Theorem 5.4 in this case. It can be generalized to the case

when X is an S^k-bundle over a compact, oriented manifold M of dimension $k+1$ and H is a top form on X. The behavior of the twisted torsion under T-duality when X is of any dimension and H is a closed 3-form remains an interesting problem. Such a relation will provide a new way of calculating twisted analytic torsions and, in particular, the classical Ray-Singer torsion using T-duality.

References

[1] M.F. Atiyah and G. Segal, Twisted K-theory and cohomology, in: P.A. Griffith (Ed.), Inspired by S.S. Chern, Nankai Tracts Math., vol. 11, World Sci. Publ., Hackensack, NJ, 2006, pp. 5-43, [arXiv:math.AT/0510674].

[2] N. Berline, E. Getzler and M. Vergne, Heat kernels and Dirac operators, Grund. Math. Wissen., 298, Springer-Verlag, Berlin, 1992.

[3] J.-M. Bismut, H. Gillet and C. Soulé, Analytic torsion and holomorphic determinant bundles. I. Bott-Chern forms and analytic torsion; II. Direct images and Bott-Chern forms; III. Quillen metrics on holomorphic determinants, Commun. Math. Phys. 115 (1988) 49-78; 79-126; 301-351.

[4] J.-M. Bismut and S. Goette, Torsions analytiques équivariantes en théorie de de Rham, C. R. Acad. Sci. Paris Sér. I Math. 332 (2001) 33-39; Equivariant de Rham torsions, Ann. Math. (2) 159 (2004) 53-216.

[5] J.-M. Bismut and K. Köhler, Higher analytic torsion forms for direct images and anomaly formulas, J. Alg. Geom. 1 (1992) 647-684.

[6] J.-M. Bismut and G. Lebeau, Immersions complexes et métriques de Quillen, C. R. Acad. Sci. Paris Sér. I Math. 309 (1989) 487-491; Complex immersions and Quillen metrics, Inst. Hautes Études Sci. Publ. Math. No. 74 (1991) 1-298.

[7] J.-M. Bismut and J. Lott, Flat vector bundles, direct images and higher real analytic torsion, J. Amer. Math. Soc. 8 (1995) 291-363.

[8] J.-M. Bismut and W. Zhang, Métriques de Reidemeister et métriques de Ray-Singer sur le déterminant de la cohomologie d'un fibré plat: une extension d'un résultat de Cheeger et Müller, C. R. Acad. Sci. Paris Sér. I Math. 313 (1991) 775-782; An extension of a theorem by Cheeger and Müller, with an appendix by F. Laudenbach, Astérisque 205 (1992) 1-235.

[9] J.-M. Bismut and W. Zhang, Milnor and Ray-Singer metrics on the equivariant determinant of a flat vector bundle, Geom. Funct. Anal. 4 (1994) 136-212.

[10] R. Bott and L. Tu, Differential forms in algebraic topology, Grad. Texts Math., vol. 82, Springer-Verlag, New York-Berlin, 1982.

[11] P. Bouwknegt, A. Carey, V. Mathai, M. Murray and D. Stevenson, Twisted K-theory and K-theory of bundle gerbes, Commun. Math. Phys. 228 (2002) 17-49, [arXiv:hep-th/0106194].

[12] P. Bouwknegt, J. Evslin and V. Mathai, T-duality: topology change from H-flux, Commun. Math. Phys. 249 (2004) 383-415, [arXiv:hep-th/0306062].

[13] M. Braverman and T. Kappeler, A refinement of the Ray-Singer torsion, C. R. Acad. Sci. Paris, Ser. I Math. 341 (2005) 497-502, [arXiv:math/0509578]; Refined analytic torsion, J. Diff. Geom. 78 (2008) 193-267, [arXiv:math/0505537].

[14] M. Braverman and T. Kappeler, Refined analytic torsion as an element of the determinant line, Geom. Topol. 11 (2007) 139-213, [arXiv:math/0510532].

[15] J. Brüning and M. Lesch, On the η-invariant of certain nonlocal boundary value problems, Duke Math. J. 96 (1999) 425-468, [arXiv:dg-ga/9609001].

[16] J. Brüning and X. Ma, An anomaly formula for Ray-Singer metrics on manifolds with boundary, C. R. Math. Acad. Sci. Paris 335 (2002) 603-608; Geom. Funct. Anal. 16 (2006) 767-837.

[17] U. Bunke, Equivariant torsion and G-CW-complexes, Geom. Funct. Anal. 9 (1999) 67-89, [arXiv:dg-ga/9711011].

[18] D. Burghelea and S. Haller, Complex-valued Ray-Singer torsion, J. Funct. Anal. 248 (2007) 27-78, [arXiv:math/0604484]; II, preprint, 2006, [arXiv:math/0610875].

[19] J. Cheeger, Analytic torsion and the heat equation, Ann. Math. (2) 109 (1979) 259-322.

[20] X. Dai and H. Fang, Analytic torsion and R-torsion for manifolds with boundary, Asian J. Math. 4 (2000) 695-714, [arXiv:math/9901052].

[21] X. Dai and R. Melrose, Adiabatic limit of the Ray-Singer analytic torsion, preprint, unpublished; X. Dai, Geometric invariants and their adiabatic limits, in: R. Greene and S.-T. Yau

(Eds.), Differential geometry: geometry in mathematical physics and related topics (Los Angeles, CA, 1990) Proc. Sympos. Pure Math., vol. 54, Part 2, Amer. Math. Soc., Providence, RI, 1993, pp. 145-156.
[22] D. Freed, Reidemeister torsion, spectral sequences, and Brieskorn spheres, J. Reine Angew. Math. 429 (1992) 75-89.
[23] D. Fried, Analytic torsion and closed geodesics on hyperbolic manifolds, Invent. Math. 84 (1986) 523-540.
[24] P.B. Gilkey, Invariance theory, the heat equation, and the Atiyah-Singer index theorem, Math. Lecture Series, vol. 11, Publish or Perish, Inc., Wilmington, DE, 1984; 2nd ed., (Studies Adv. Math.), CRC Press, Boca Raton, FL, 1995.
[25] G. Grubb, Spectral boundary conditions for generalizations of Laplace and Dirac operators, Commun. Math. Phys. 240 (2003) 243-280, [arXiv:math/0302286].
[26] G. Grubb, A resolvent approach to traces and zeta Laurent expansions, in: B. Booß-Bavnbek, G. Grubb and K.P. Wojciechowski (Eds.), Spectral geometry of manifolds with boundary and decomposition of manifolds, (Roskilde Univ., 2003), Contemp. Math., vol. 366, Amer. Math. Soc., Providence, RI, 2005, pp. 67-93, corrected in [arXiv:math/0311081v4].
[27] G. Grubb and R.T. Seeley, Weakly parametric pseudodifferential operators and Atiyah-Patodi-Singer boundary problems, Invent. Math. 121 (1995) 481-529.
[28] M. Gualtieri, Generalized complex geometry, Oxford University DPhil thesis, 2003, [arXiv:math/0401221].
[29] M. Gualtieri, Generalized geometry and the Hodge decomposition, preprint, 2004, [arXiv:math/0409093].
[30] V. Guillemin, A new proof of Weyl's formula on the asymptotic distribution of eigenvalues, Adv. Math. 55 (1985) 131-160.
[31] N. Hitchin, Generalized Calabi-Yau manifolds, Quart. J. Math. 54 (2003) 281-308, [arXiv:math/0209099].
[32] F.F. Knudsen and D. Mumford, The projectivity of the moduli space of stable curves. I. Preliminaries on "det" and "Div", Math. Scand. 39 (1976) 19-55.
[33] M. Kontsevich and S. Vishik, Determinants of elliptic pseudo-differential operators, preprint, 1994, [arXiv:hep-th/9404046]; Geometry of determinants of elliptic operators, in S. Gindikin, J. Lepowsky and R.L. Wilson (Eds.), Functional analysis on the eve of the 21st century, vol. 1 (New Brunswick, NJ, 1993), Prog. Math., vol. 131, Birkhäuser, Boston, MA, 1995, pp. 173-197, [arXiv:hep-th/9406140].
[34] J. Lott and M. Rothenberg, Analytic torsion for group actions, J. Diff. Geom. 34 (1991) 431-481.
[35] W. Lück, Analytic and topological torsion for manifolds with boundary and symmetry, J. Diff. Geom. 37 (1993) 263-322.
[36] W. Lück, T. Schick and T. Thielmann, Torsion and fibrations, J. Reine Angew. Math. 498 (1998) 1-33, [arXiv:dg-ga/9707010].
[37] X. Ma, Formes de torsion analytique et familles de submersions, C. R. Acad. Sci. Paris Sér. I Math. 324 (1997) 205-210; I., Bull. Soc. Math. France 127 (1999) 541-621; II., Asian J. Math. 4 (2000) 633-667.
[38] X. Ma, Functoriality of real analytic torsion forms, Israel J. Math. 131 (2002) 1-50.
[39] S. Maumary, Contributions à la théorie du type simple d'homotopie, Comment. Math. Helv. 44 (1969) 410-437.
[40] W. Müller, Analytic torsion and R-torsion of Riemannian manifolds, Adv. Math. 28 (1978) 233-305.
[41] W. Müller, Analytic torsion and R-torsion for unimodular representations, J. Amer. Math. Soc. 6 (1993) 721-753.
[42] S.P. Novikov, Multivalued functions and functionals. An analogue of the Morse theory, Dokl. Akad. Nauk SSSR 260 (1981) 31-35; English translation: Soviet Math. Dokl. 24 (1981) 222–226.
[43] S.P. Novikov, Bloch homology. Critical points of functions and closed 1-forms, Dokl. Akad. Nauk SSSR 287 (1986) 1321-1324; English translation: Soviet Math. Dokl. 33 (1986) 551-555.
[44] A.V. Pazhitnov, An analytic proof of the real part of Novikov's inequalities, Dokl. Akad. Nauk SSSR 293 (1987) 1305-1307; English translation: Soviet Math. Dokl. 35 (1987) 456-457.
[45] D. Quillen, Determinants of Cauchy-Riemann operators on Riemann surfaces, Funct. Anal. Appl. 19 (1985) 31-34.

[46] D.B. Ray, Reidemeister torsion and the Laplacian on lens spaces, Adv. Math. 4 (1970) 109-126.
[47] D.B. Ray and I.M. Singer, R-torsion and the Laplacian on Riemannian manifolds, Adv. Math. 7 (1971) 145-210.
[48] D.B. Ray and I.M. Singer, Analytic torsion, in: D.C. Spencer (Ed.), Partial differential equations (Univ. California, Berkeley, Calif., 1971), Proc. Sympos. Pure Math., vol. XXIII, Amer. Math. Soc., Providence, R.I., 1973, pp. 167-181.
[49] D.B. Ray and I.M. Singer, Analytic torsion for complex manifolds, Ann. Math. (2) 98 (1973) 154-177.
[50] R. Rohm and E. Witten, The antisymmetric tensor field in superstring theory, Ann. Phys. 170 (1986) 454-489.
[51] A.S. Schwarz, The partition function of degenerate quadratic functional and Ray-Singer invariants, Lett. Math. Phys. 2 (1977/78) 247-252; The partition function of a degenerate functional, Commun. Math. Phys. 67 (1979) 1-16.
[52] R.T. Seeley, Complex powers of an elliptic operator, in: A.P. Calderón (Ed.), Singular Integrals (Chicago, IL, 1966), Proc. Symp. Pure Math., vol. X, Amer. Math. Soc., Providence, RI, 1967, pp. 288-307.
[53] M.A. Shubin, Pseudodifferential operators and spectral theory, Springer-Verlag, Berlin, 1987; 2nd ed., 2001.
[54] G. Su and W. Zhang, A Cheeger-Müller theorem for symmetric bilinear torsions, Chin. Ann. Math. 29B (2008) 385-424, [arXiv:math/0610577].
[55] S.M. Vishik, Generalized Ray-Singer conjecture. I. a manifold with a smooth boundary, Commun. Math. Phys. 167 (1995) 1-102, [arXiv:hep-th/9305184].
[56] M. Wodzicki, Local invariants of spectral asymmetry, Invent. Math. 75 (1984) 143-177.

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